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Technical Report No. 2

SELECTION OF A DELAY LINE MODEL

by

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and

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to

Office of Naval Research
Department of the Navy

Contract Nonr-1100(20)

School of Mechanical Engineering
Purdue University

April, 1964

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438258

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SELECTION OF A DELAY LINE MODEL

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A mathematical model of a linear system can be derived using an approximation of the convolution integral. This paper selects the model such that responses of the model to commonly occurring inputs are closest to corresponding responses of the system. The transfer function of the model is the product of the system transfer function by a linear combination of two delay terms divided by an infinite product. If the model delay time is small the infinite product may be replaced by 1, and thus may be dropped. The delay time and the number of delay elements are selected such that the responses of the simplified model are closest to corresponding responses of the system. The validity of the simplification is investigated for various inputs by comparing the responses of the simplified model with those of the exact model. It is found for several types of commonly occurring inputs that the number of delay elements should be chosen as large

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as physically possible. The results show that the value of the delay time should be selected as a function of the number of delay elements and of the system bandwidth. It is further shown that this function is the same for each of the inputs.

INTRODUCTION

To design a controller for a physical plant, the system must be identified by some method. An analytical representation of the system which may be used to find the output of the system from its input is often desired. Truxal¹ presents techniques for evaluating the differential equation, the frequency response and the impulse response of linear, time-invariant systems.

In 1957 Goodman and Reswick² presented an application of the theory developed by Tustin³, Lewis⁴ and others as a means of identification for linear, time-invariant systems. A mathematical model of the system is derived by approximating the convolution integral with a weighted time series having a delay interval T . The smaller the value of T and the larger the number of elements retained in the time series, the better is the approximation. Given a system with unknown characteristics the impulse response may be approximated from the time series as closely as desired.

The delay line synthesizer of Goodman and Reswick² (See Figure 1.) constitutes a physical model which uses only a finite number of elements in the time series. The weighting factors are determined by adjusting the synthesizer settings until an optimum fit is obtained between the model and system responses to normal operating inputs. Chang⁵ continued the work in applying the delay line synthesizer.

In this paper the weighted time series is used to derive a mathematical model by keeping only a physically realizable number of elements in the series. The key to this study is the use of an infinite product employed by R. Oldenburger and R. E. Goodson⁶ for transcendental functions arising in the analysis of fluid flow through pipes. The Laplace transform of the terms in the series is taken and a transfer function obtained by writing the transformation in closed form. By choosing a small value of delay time the infinite product in the transfer function may be neglected and a simplified model representation obtained. This approximate model representation is used to compare system and model responses to various inputs. From this comparison the number of delay elements in the model and the value of the delay time are selected such that the responses of the model are closest to corresponding responses of the system. The exact model responses are compared with the approximate responses to show that the simplification is justified.

In what follows t denotes time.

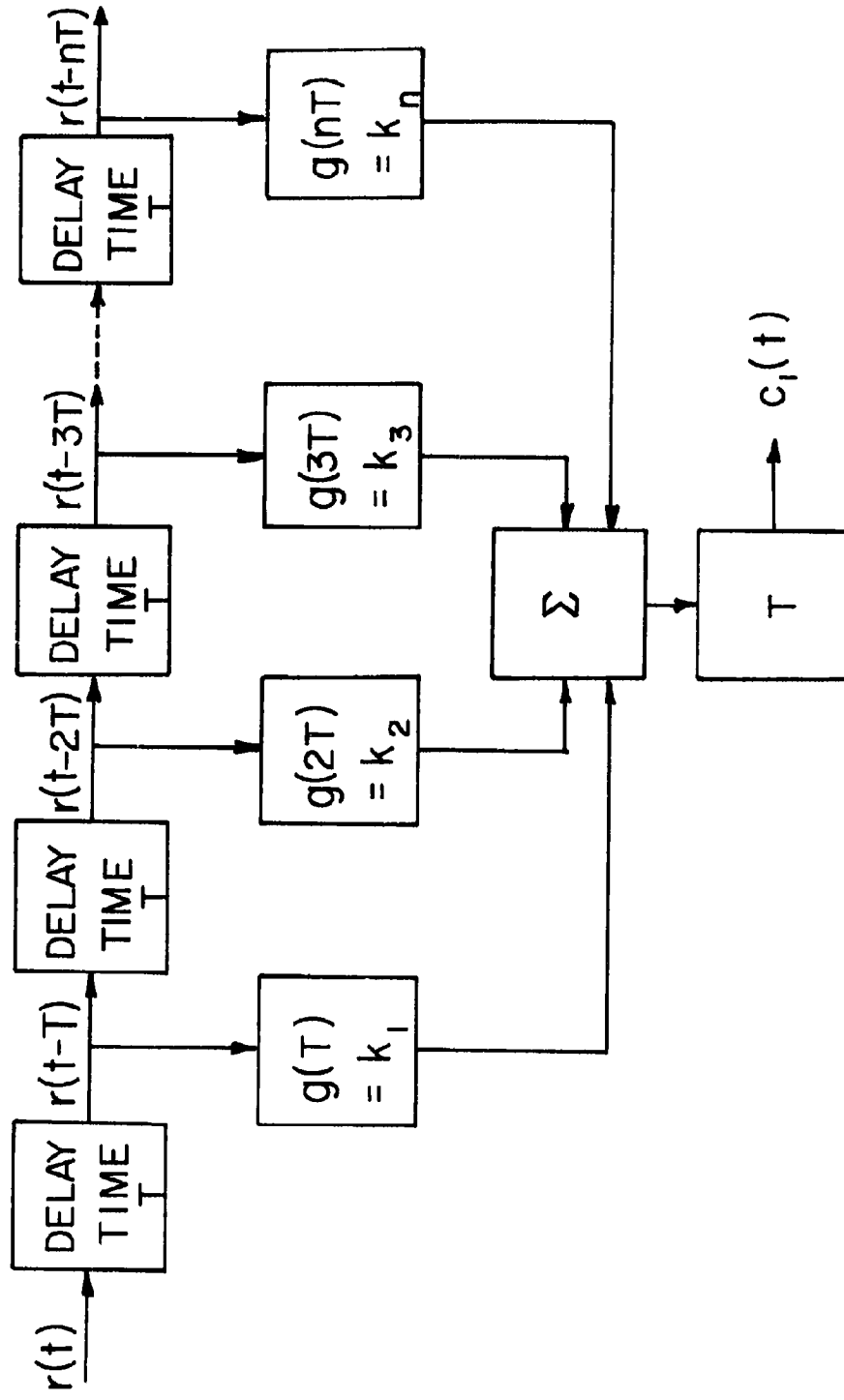


Figure 1
Block Diagram of a Delay Line Synthesizer

1.

DERIVATION OF THE DELAY LINE MODEL

Let $g(t)$ represent the impulse response, or weighting function, of a linear system. The response $c(t)$ of the system to an input $r(t)$ may be determined by the convolution integral⁷ as in

$$c(t) = \int_0^t g(\sigma) r(t - \sigma) d\sigma, \quad t \geq 0 \quad (1)$$

where

$$r(t - \sigma) = 0, \quad t < \sigma \quad (2)$$

Let $c_1(t)$ be defined by

$$\begin{aligned} c_1(t) = & [g(T) r(t - T) + g(2T) r(t - 2T) + \dots \\ & + g(nT) r(t - nT)]T \end{aligned} \quad (3)$$

where T is an increment of the variable σ and n is a positive integer.

If T is small and n is large enough we may write

$$c(t) \approx c_1(t) \quad (4)$$

where \approx means "is approximately equal to." We designate $c_1(t)$ as the response of the delay line model (DIM) of the system. The Laplace transform of a function of time $f(t)$, that is $\mathcal{L}[f(t)]$, is denoted by $F(s)$, where s is the Laplace variable. From Equation (1) it may be shown that⁸

$$C(s) = R(s) G(s) \quad (5)$$

Since r is defined by Equation (2) we have⁸

$$\mathcal{L}[r(t - nT)] = R(s) e^{-nTs} \quad (6)$$

Thus, taking the Laplace transform of the terms in Equation (3) yields

$$C_1(s) = R(s) G_1(s) \quad (7)$$

where

$$G_1(s) = [g(T) e^{-Ts} + g(2T) e^{-2Ts} + \dots + g(nT) e^{-nTs}] T \quad (8)$$

Let $G_1(s)$ be defined as the DLM transfer function. Suppose that a linear system has an unknown transfer function $G(s)$. If the impulse response $g(t)$ is measured at times $T, 2T, \dots, nT$, then $G_1(s)$ is determined. The smaller the delay time T and the larger the number of delay elements n , the better is the approximation in Relation (3), or the better does $G_1(s)$ approximate $G(s)$.

There must exist a lower limit for T and an upper bound for n if the DLM is to be physically realizable. Knowledge of how the difference between system and model responses depends on n and T is desired. These parameters are to be selected such that the model fulfills nominal engineering requirements.

2.

FIRST ORDER SYSTEM

Consider the system described by the transfer function

$$G(s) = \frac{a}{s + a} \quad (9)$$

Employing the inverse Laplace transform we obtain the weighting function

$$g(t) = a e^{-at} \quad (10)$$

Hence $G_1(s)$ in Equation (8) becomes

$$G_1(s) = a T \left[e^{-T(a+s)} + e^{-2T(a+s)} + \dots + e^{-nT(a+s)} \right] \quad (11)$$

Using the transformation

$$z = T(a + s) \quad (12)$$

we write Equation (11) in the closed form

$$G_1\left(\frac{z - aT}{T}\right) = a T \left[\frac{e^{-\frac{1}{2}z} - e^{-(n+\frac{1}{2})z}}{e^{\frac{1}{2}z} - e^{-\frac{1}{2}z}} \right] \quad (13)$$

It is desired to expand the denominator of the term on the right of equation (13) as an infinite product. Let a function $Z(z)$ have simple zeros at the points z_1, z_2, z_3, \dots , where

$$\lim_{k \rightarrow \infty} |z_k| = \infty$$

these being the only zeros of $Z(z)$ and $z_k \neq 0$. If $Z(z)$ is analytic for all values of z , it may be shown that⁹

$$Z(z) = Z(0) \left\{ \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{z_k}\right) \exp\left(\frac{z}{z_k}\right) \right] \right\} \exp\left[\frac{Z'(0)}{Z(0)} z\right] \quad (14)$$

where $\exp [\quad]$ denotes raising the base "e" to the power indicated in the brackets. The zeros of the denominator above are given by

$$z_k = 2k \pi j, \quad k = 0, \pm 1, \pm 2, \dots \quad (15)$$

where $j = \sqrt{-1}$. Hence the conditions for expansion of the denominator of Equation (13) by Equation (14) are not satisfied due to the root $z_0 = 0$. This problem is solved by writing

$$e^{\frac{1}{2}z} - e^{-\frac{1}{2}z} = z \left[\frac{e^{\frac{1}{2}z} - e^{-\frac{1}{2}z}}{z} \right] \quad (16)$$

The bracketed expression now satisfies the conditions for expansion, the zeros, z_r , being given by

$$z_r = 2r \pi j, \quad r = \pm 1, \pm 2, \dots \quad (17)$$

Thus by Equations (14) and (16)

$$e^{\frac{1}{2}z} - e^{-\frac{1}{2}z} = z \left[1 + \frac{z^2}{4 \pi^2} \right] \left[1 + \frac{z^2}{16 \pi^2} \right] \dots \quad (18)$$

Substituting the expression for z in Equation (13) yields

$$G_1(s) = \left[\frac{a}{s + a} \right] \left[\prod_s \right] H(s) \quad (19)$$

where

$$\prod_s = \prod_{m=1}^{\infty} \left\{ \frac{1}{\left[1 + \frac{T^2(a+s)^2}{4 \prod^2 \omega^2} \right]} \right\} \quad (20)$$

$$H(s) = \left[e^{-\frac{1}{2}aT} e^{-\frac{1}{2}Ts} - e^{-a(n+\frac{1}{2})T} e^{-(n+\frac{1}{2})Ts} \right] \quad (21)$$

and the index m is a non-negative integer. Figure 2 presents a block diagram of $G_1(s)$ in Equation (19).

Let a new function $C_1^*(s)$ be given by

$$C_1^*(s) = R(s) \left[\frac{a}{s+a} \right] \prod_s \quad (22)$$

Then from Equations (7) and (19) it follows that

$$C_1(s) = C_1^*(s) H(s) \quad (23)$$

Substituting the expression for $H(s)$ and employing the inverse Laplace transform gives

$$\begin{aligned} c_1(t) = & e^{-\frac{1}{2}aT} c_1^* \left[t - \frac{1}{2}T \right] u \left[t - \frac{1}{2}T \right] \\ & - e^{-a(n+\frac{1}{2})T} c_1^* \left[t - (n + \frac{1}{2})T \right] u \left[t - (n + \frac{1}{2})T \right] \end{aligned} \quad (24)$$

where $c_1^*(t)$ is the inverse Laplace of $C_1^*(s)$ and the function u is defined by

$$u(t - t_1) = \begin{cases} 0, & t < t_1 \\ 1, & t \geq t_1 \end{cases} \quad t_1 \geq 0 \quad (25)$$

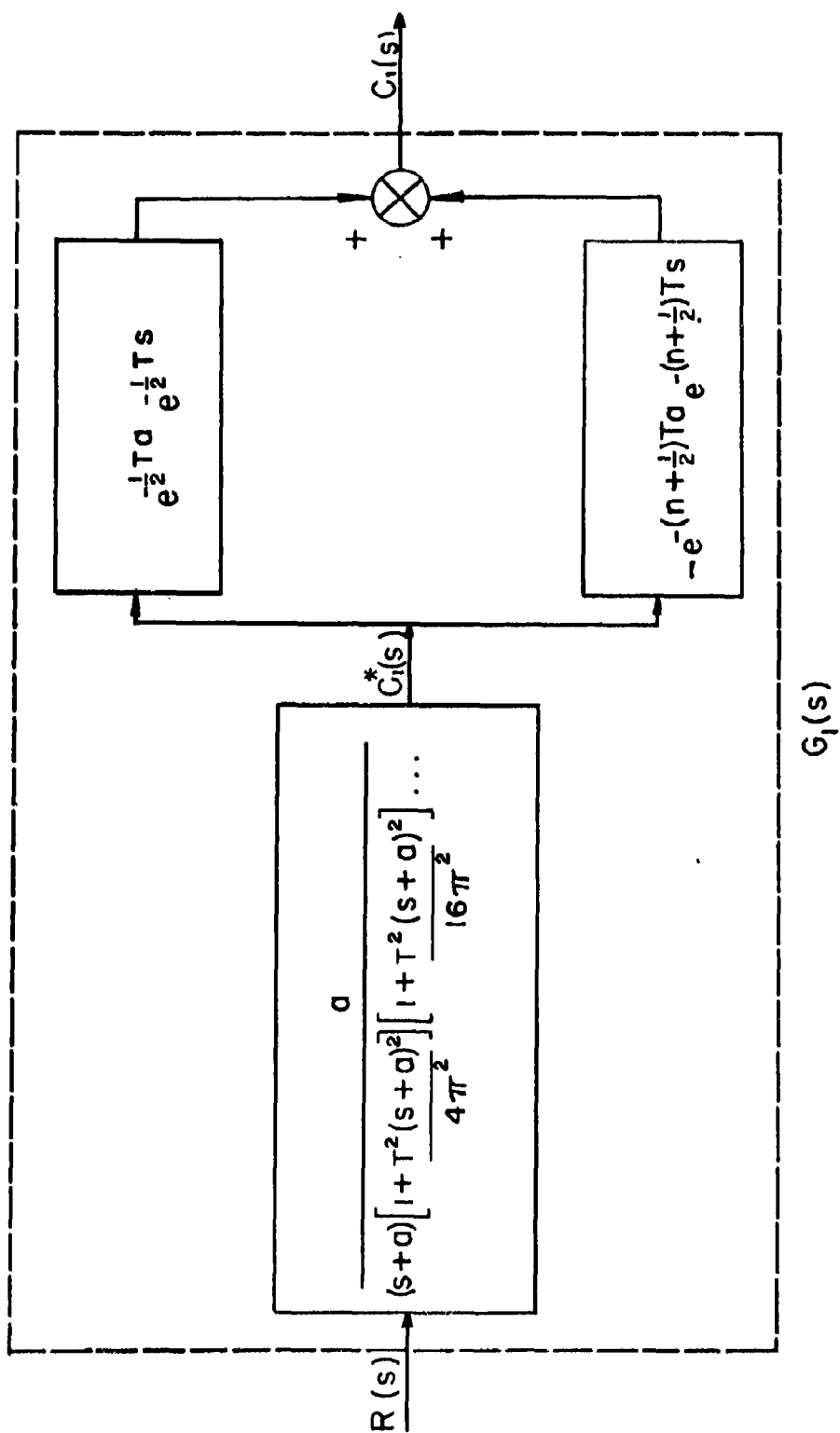


Figure 2
Block Diagram of Exact Model Transfer Function

3. APPROXIMATION OF THE DLM TRANSFER FUNCTION

Let T be small enough such that the product aT is much less than one ($aT \ll 1$). Then a reasonable approximation of $G_1(s)$ results when the infinite product \prod_g is neglected. From Equation (22)

$$C_1^*(s) \approx C(s), \quad aT \ll 1 \quad (26)$$

We introduce $C_{1A}(s)$, where

$$C_{1A}(s) = C(s) H(s) \quad (27)$$

Hence from Equation (23)

$$C_1(s) \approx C_{1A}(s), \quad aT \ll 1 \quad (28)$$

Let the approximate model response be denoted by $c_{1A}(t)$, which is the inverse Laplace transform of $C_{1A}(s)$. Substituting the expression for $H(s)$ in Equation (21) and employing the inverse Laplace transform we obtain

$$\begin{aligned} c_{1A}(t) = & e^{-\frac{1}{2}aT} c\left[t - \frac{1}{2}T\right] u\left[t - \frac{1}{2}T\right] \\ & - e^{-a(n+\frac{1}{2})T} c\left[t - (n + \frac{1}{2})T\right] u\left[t - (n + \frac{1}{2})T\right] \end{aligned} \quad (29)$$

To differentiate from the approximate model response we henceforth refer to $c_1(t)$ as the exact model response. The problem of selecting n and T

is simplified using Relation (28). In the following study we select the model parameters and show that employing this relation is reasonable for five commonly occurring inputs to the system.

4.

STEP INPUT

Consider a step input to the model and the system given by

$$r(t) = u(t) \quad . \quad (30)$$

Using the Laplace transform, from Equation (5) we obtain the system response as

$$c(t) = [1 - e^{-at}] u(t) \quad . \quad (31)$$

From Equation (29) the approximate model response becomes

$$c_{1A}(t) = 0 \quad , \quad 0 \leq t < \frac{1}{2}T \quad (32a)$$

$$c_{1A}(t) = e^{-\frac{1}{2}aT} - e^{-at} \quad , \quad \frac{1}{2}T \leq t < (n + \frac{1}{2})T \quad (32b)$$

$$c_{1A}(t) = e^{-\frac{1}{2}aT} [1 - e^{-anT}] \quad , \quad t \geq (n + \frac{1}{2})T \quad . \quad (32c)$$

Selection of Model Parameters

By Relations (32) the approximate model response varies with time only for $\frac{1}{2}T \leq t < (n + \frac{1}{2})T$. For t outside these limits the response of the model fails to follow that of the system. Figures 3 through 5 show the system and model responses to a step input of magnitude K with unity bandwidth ($a = 1$) and various values of T and n . The responses are normalized by dividing each term by K . The model response is more satisfactory for large n and small T , as for example, $n = 300$ and $T = 0.01$ in

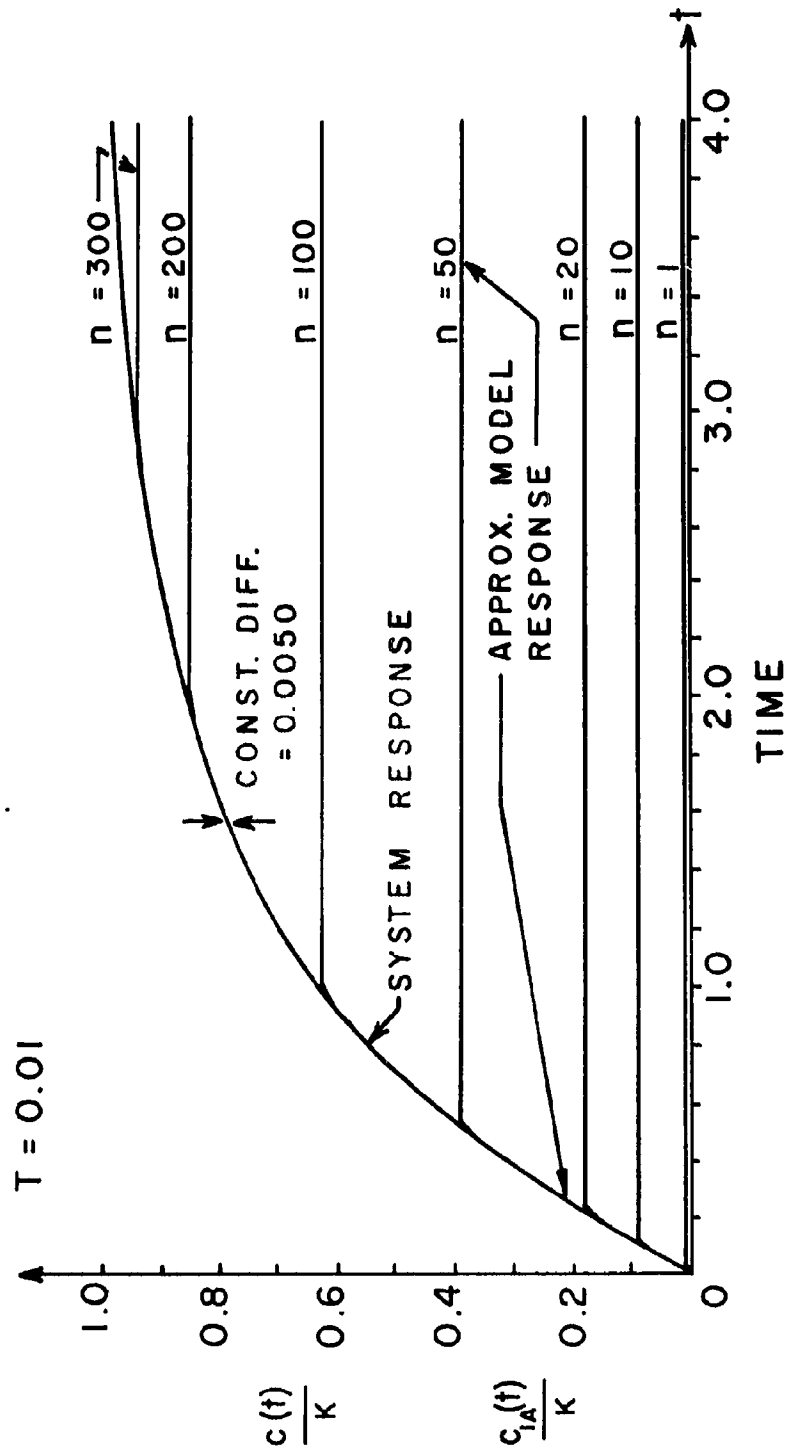


Figure 3
System and Approximate Model Responses to a
Step Input for $T = 0.01$

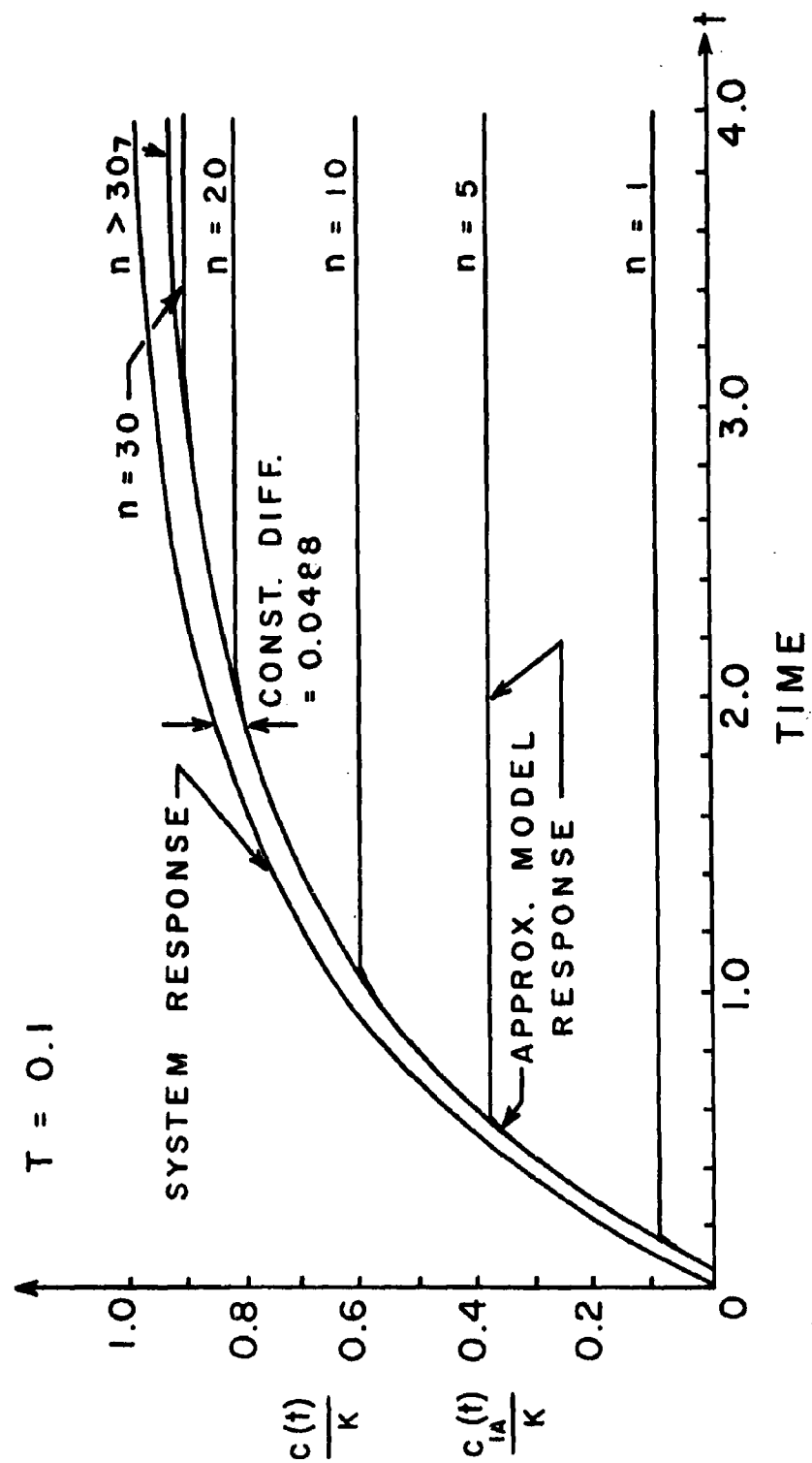


Figure 4
System and Approximate Model Responses to a
Step Input for $T = 0.1$

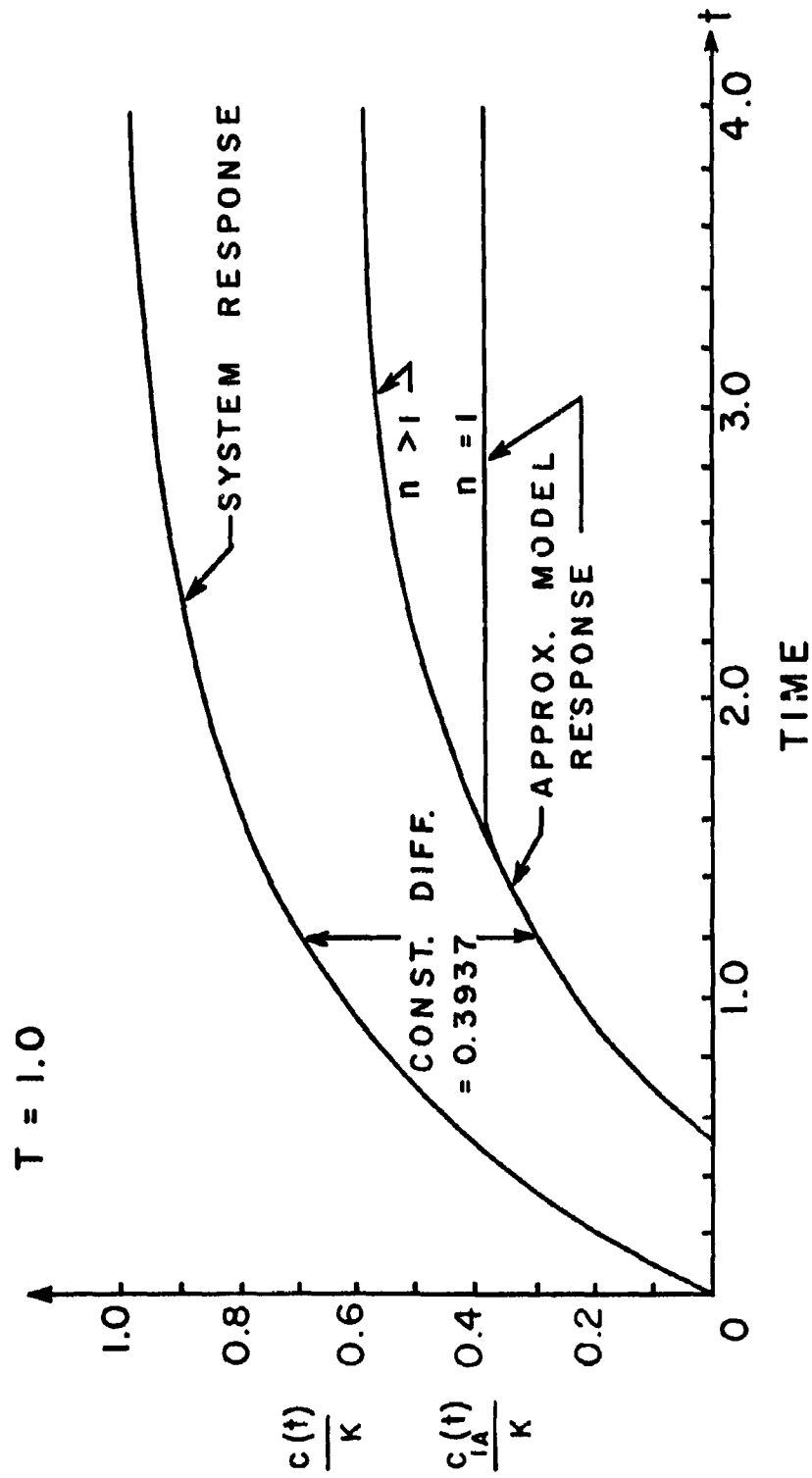


Figure 5
System and Approximate Model Responses to a
Step Input for $T = 1.0$

Figure 3. However, a delay line model with such parameters would be unpractical. Goodman and Reswick use $n = 20$ in the delay line synthesizer². We see from the response plots mentioned above that for any n the final value of $c_{1A}(t)$ has a maximum which is dependent on T . We designate T_0 as that value of T for which the maximum final value occurs. As $T \rightarrow T_0$ from values less than T_0 , the final value of $c_{1A}(t)$ increases, but the model follows the system less closely for the range $\frac{1}{2}T \leq t < (n + \frac{1}{2})T$. For finite non-zero values of n and T the model output is always less than that of the system. We choose to select n and T to maximize the final value of $c_{1A}(t)$. Since for any T the best n to accomplish this is infinitely large, we first select n as large as reasonably possible for applications. The time T is then chosen to maximize the final value, V , of the approximate model response, where from Equation (32c)

$$V = e^{-\frac{1}{2}aT} [1 - e^{-anT}] \quad (33)$$

Let n_0 be the value of n selected for the model. The maximum value of V occurs when $T = T_0$, where

$$T_0 = \frac{\ln(2n_0 + 1)}{an_0} \quad (34)$$

The maximum final value, V_M , of $c_{1A}(t)$ is obtained by replacing T in Equation (33) by the expression for T_0 , giving

$$V_M = (2n_0 + 1)^{-\frac{1}{2n_0}} [1 - (2n_0 + 1)^{-1}] \quad (35)$$

If $n_0 = 20$, the value used by Goodman and Reswick, and $a = 1$, for

simplicity, Equations (34) and (35) yield

$$T_0 = 0.1857, \quad V_M = 0.8891$$

Figure 6 indicates system and approximate model normalized responses for a step input of magnitude K , with $a = 1$ and $T = T_0$ at $n_0 = 20$. It is easily verified from Equations (28) and (29) that

$$\lim_{n_0 \rightarrow \infty} T_0 = 0, \quad (36)$$

$$\lim_{n_0 \rightarrow \infty} V_M = 1. \quad (37)$$

Thus better model response, using values of T_0 , is obtained by increasing n .

Comparison of the Approximate Model with the Exact Model

We have selected the model parameters using an approximation which neglects the infinite product \prod_s in the expression for $G_1(s)$. To determine the effects of using this approximation we replace $R(s)$ in Equation (22) by the Laplace transform of the step input to obtain

$$C_1^*(s) = \left[\frac{1}{s} \right] \left[\frac{a}{s + a} \right] \prod_s. \quad (38)$$

Expanding this expression for $C_1^*(s)$ in partial fractions and employing the inverse Laplace transform, we have

$$c_1^*(t) = \prod_T - e^{-at} \left\{ 1 + \sum_{m=1}^{\infty} A_m \sin \left[\frac{2\prod_m}{T} t - \theta_m \right] \right\} \quad (39)$$

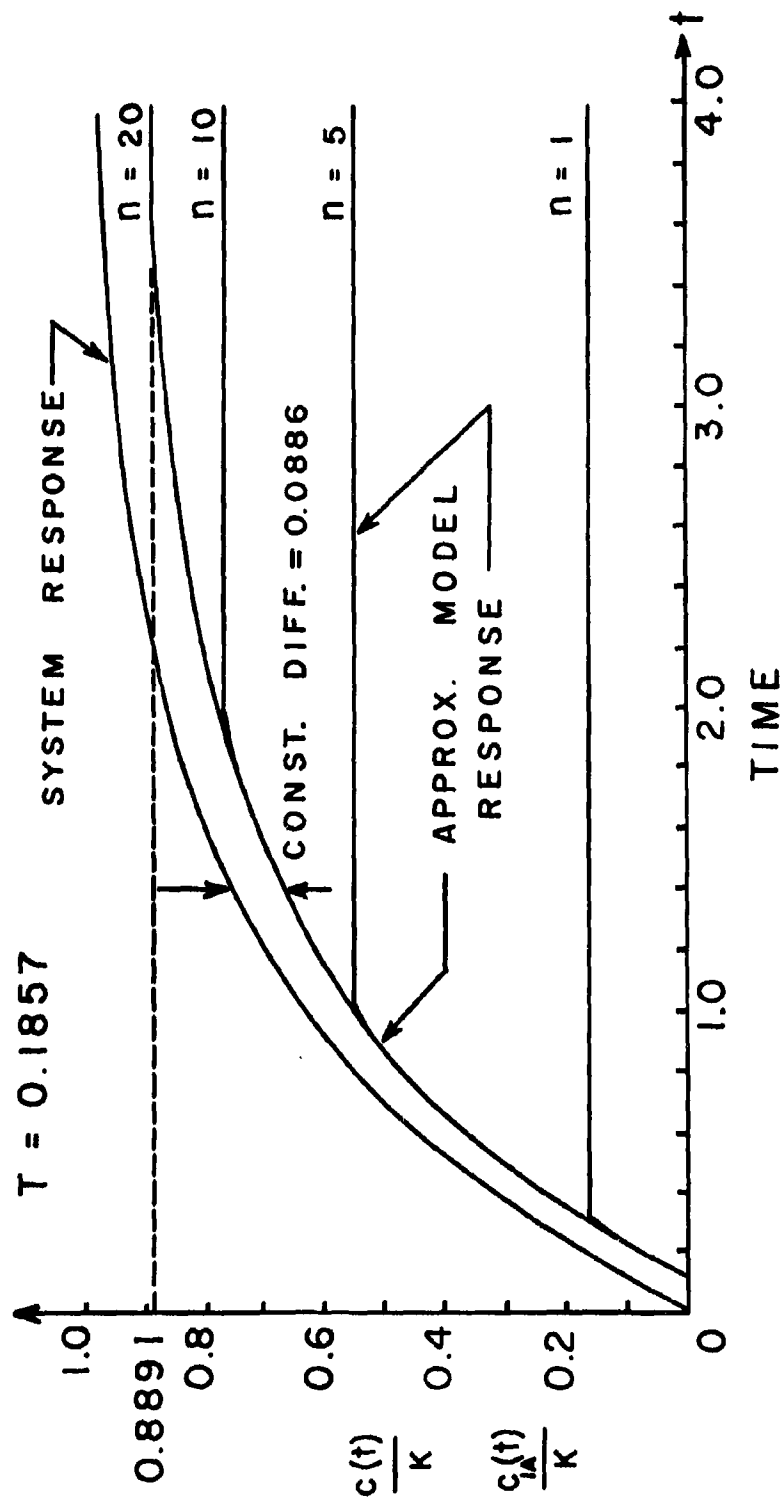


Figure 6
System and Approximate Model Responses to a Step
Input for $T = T_0$ at $n_0 = 20$

where

$$\prod_T = \frac{1}{\prod_{m=1}^{\infty} \left[1 + \frac{a^2 T^2}{4 \pi^2 m^2} \right]} , \quad (40)$$

$$A_m = \frac{\prod_m}{\alpha_m} , \quad (41)$$

$$\theta_m = \tan^{-1} \beta_m , \quad (42)$$

$$\alpha_m = \sqrt{1 + \frac{4 \pi^2 m^2}{a^2 T^2}} , \quad (43)$$

$$\beta_m = \frac{a T}{2 \pi m} , \quad (44)$$

$$\prod_m = \left\{ \prod_{i=1}^{m-1} \left[\frac{1}{1 - \left(\frac{m}{i}\right)^2} \right] \right\} \left\{ \prod_{i=m+1}^{\infty} \left[\frac{1}{1 - \left(\frac{m}{i}\right)^2} \right] \right\} . \quad (45)$$

Hence from Equation (24)

$$c_1(t) = 0 , \quad 0 \leq t < \frac{1}{2}T \quad (46a)$$

$$c_1(t) = \left[\prod_T \right] e^{-\frac{1}{2}aT} - e^{-at} \left\{ 1 + \sum_{m=1}^{\infty} A_m \left[\sin \frac{2 \pi m}{T} t - \theta_m - m\pi \right] \right\} ,$$

$$\frac{1}{2}T \leq t < (n + \frac{1}{2})T \quad (46b)$$

$$c_1(t) = \left[\prod_T \right] \left[1 - e^{-aT} \right] e^{-\frac{1}{2}aT}, \quad t \geq (n + \frac{1}{2})T. \quad (46c)$$

Comparison of Equations (46) with Equations (32) indicates that both $c_1(t)$ and $c_{1A}(t)$ are zero for $0 \leq t < \frac{1}{2}T$. For the second range, $\frac{1}{2}T \leq t < (n + \frac{1}{2})T$, the effect of the approximation is to omit the attenuating infinite product \prod_T in the first term and the infinite sinusoidal series in the second term of the exact model response. For $t \geq (n + \frac{1}{2})T$ the approximation omits \prod_T in the exact model final value. We note that $\prod_T \rightarrow 1$ as $T \rightarrow 0$. Further, $\prod_m = (-1)^{m+1} 2$. For small aT we have $\alpha_m \approx \frac{2\prod_m}{aT}$, $\phi_m \approx \alpha_m^{-1} \approx 0$. Taking $\phi_m = 0$, $\alpha_m = \frac{2\prod_m}{aT}$ the summation in Equation (39) is¹³ between zero and $\frac{aT}{2}$ depending on the value of t . The same is true for the summation in Equation (46b). These results and numerical studies indicate that if $aT < \frac{1}{2}$ one can neglect the summation and hence the product \prod_s .

To evaluate graphically the effects of the approximation we employ Equation (3) for the exact model response. Let the inverse Laplace transform $f(t)$ of a function $F(s)$ be denoted by $\mathcal{L}^{-1} [F(s)]$. Since⁸

$$\mathcal{L}^{-1} \left[\left(\frac{1}{s} \right) (e^{-nTs}) \right] = u(t - nT), \quad (47)$$

Equations (3) and (10) are used to give

$$\begin{aligned} c_1(t) = aT & \left[e^{-aT} u(t - T) + e^{-2aT} u(t - 2T) + \dots \right. \\ & \left. + e^{-naT} u(t - nT) \right] \end{aligned} \quad (48)$$

Figure 7 shows the system and exact model normalized responses for a step input of magnitude K , with $a = 1$, $T = T_0$ and $n_0 = 20$. Comparing these plots with those in Figures 3 through 6 indicates that the most significant effect of the approximation is the smoothing of the "staircase"

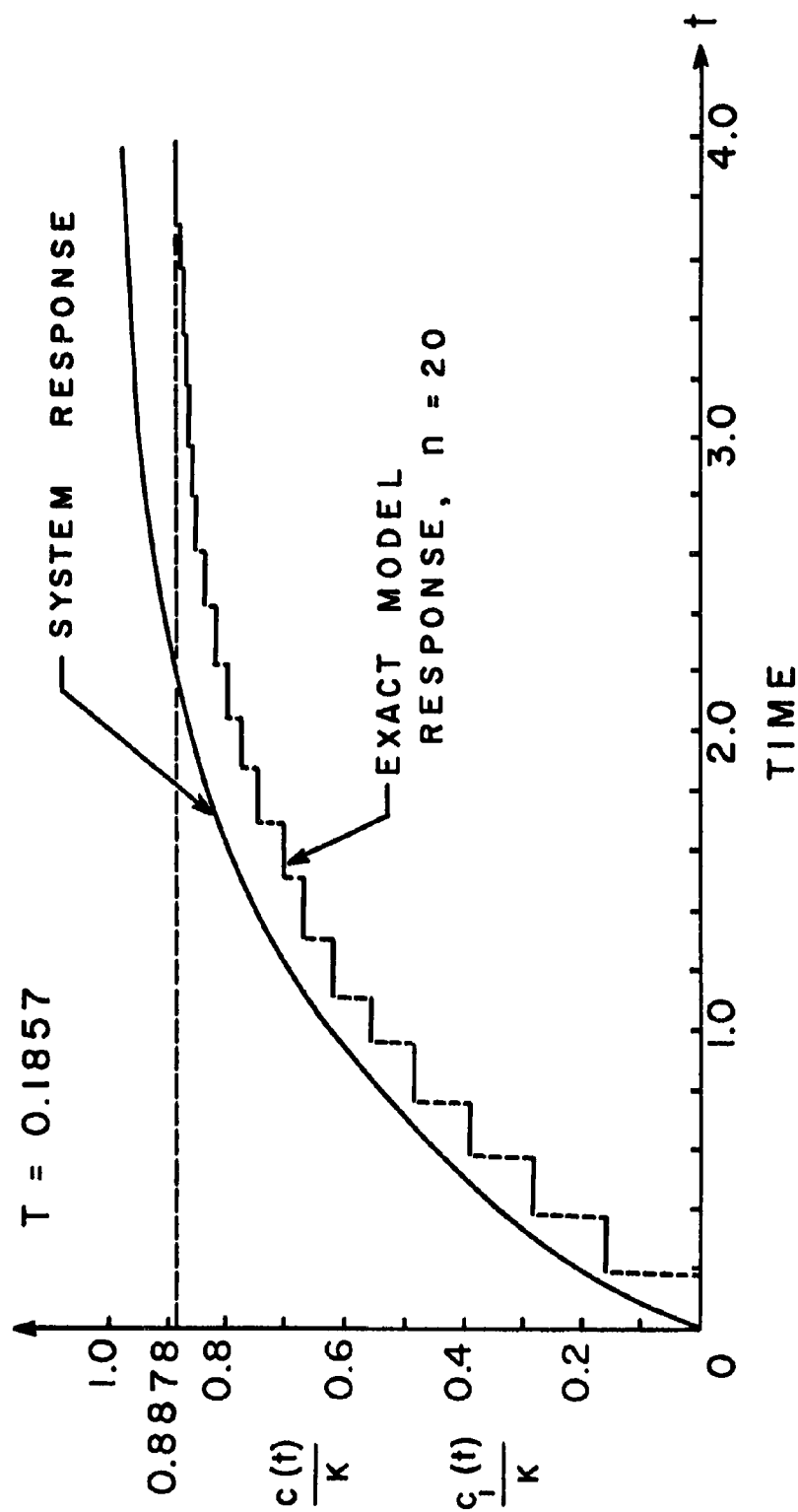


Figure 7
System and Exact Model Responses to a Step
Input for $T = T_0$ at $n_0 = 20$

response of the exact model. A further effect of the approximation is to decrease the time at which the time-variant model output begins from T to $\frac{1}{2}T$. The time at which it ends is increased from nT to $(n + \frac{1}{2})T$. This as well as the smoothing is attributed to omitting the sinusoidal series in the approximate model response.

We wish to determine whether the results in selecting the approximate model are applicable to the exact model. It is recalled that n was chosen as large as reasonably possible for applications. The delay time T was then selected such that the final value of the approximate model response given by Equation (33) was maximized. From Equation (46c) the exact model final value, V_1 , is given by

$$V_1 = \left[\prod_T \right] \left[1 - e^{-anT} \right] e^{-\frac{1}{2}aT} \quad (49)$$

An attempt to maximize this expression for V_1 by differentiating with respect to T would be fruitless. We choose to select n and T for the exact model the same as for the approximate model provided that the value of \prod_T is approximately unity. This means that the product aT must be small. It is desired to know how small. In his discussion of the paper by Goodman and Reswick², C. M. Chang notes that if the delay time T is greater than $\frac{1}{2a}$, where a is the system bandwidth, the frequency response, particularly in the high frequency region, obtained on the delay line synthesizer is questionable. This is derived from the fundamental theorem of sampling¹⁰ which states that a signal is completely determined by values of the signal (samples) taken at a series of instants separated by $T_1 = \frac{1}{2a}$, where T_1 is the sampling period. From these considerations, for the remainder of this work it will be assumed that

$$0 < a T < \frac{1}{2} \quad . \quad (50)$$

Let new quantities Y_m and W_m be defined by

$$Y_m = \frac{a^2 T^2}{4 \pi^2 m^2} \quad , \quad (51)$$

$$W_m = \sum_{m=1}^{\infty} \ln [1 + Y_m] \quad . \quad (52)$$

Then for the denominator of the expression for \prod_T in Equation (40) we have

$$\prod_{m=1}^{\infty} [1 + Y_m] = e^{W_m} \quad . \quad (53)$$

But it is seen that

$$W_m < \sum_{m=1}^{\infty} [Y_m] \quad , \quad Y_m > 0 \quad . \quad (54)$$

Hence from the relation¹¹

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} \quad (55)$$

we may write

$$W_m < \frac{1}{24} \quad , \quad 0 < a^2 T^2 < \frac{1}{4} \quad . \quad (56)$$

This gives

$$\frac{1}{e^{W_m}} > 0.9900 \quad (57)$$

and we conclude from Equation (40) that

$$0.9900 < \overline{\Pi}_T < 1 \quad . \quad (58)$$

Consequently it is safe to assume that the results in selecting the approximate model are reasonably applicable to the exact model.

5.

RAMP INPUT

The ramp signal given by

$$r(t) = t u(t) \quad (59)$$

is the second input to be considered. Using transform techniques we obtain from Equation (5)

$$c(t) = \left[t - \frac{1}{a} + \frac{1}{a} e^{-at} \right] u(t) \quad (60)$$

To derive the approximate model response we substitute this expression for $c(t)$ in Equation (29), giving

$$c_{1A}(t) = 0, \quad 0 \leq t < \frac{1}{2}T \quad (61a)$$

$$c_{1A}(t) = e^{-\frac{1}{2}aT} \left[t - \frac{1}{2}T - \frac{1}{a} \right] + \frac{1}{a} e^{-at}, \quad \frac{1}{2}T \leq t < (n + \frac{1}{2})T \quad (61b)$$

$$c_{1A}(t) = \left[e^{-\frac{1}{2}aT} - e^{-a(n+\frac{1}{2})T} \right] t - e^{-\frac{1}{2}aT} \left\{ \frac{1}{2}T + \frac{1}{a} - e^{-anT} \left[(n + \frac{1}{2})T + \frac{1}{a} \right] \right\}, \quad t \geq (n + \frac{1}{2})T \quad (61c)$$

Selection of Model Parameters

Comparison of the approximate model response with the system response indicates that for small T the response of the model follows that of the system reasonably well for the second range, $\frac{1}{2}T \leq t < (n + \frac{1}{2})T$. For the first period, $0 \leq t < \frac{1}{2}T$, the approximate model output is zero, as in the step input case, while for the last range, $t \geq (n + \frac{1}{2})T$, a lag in slope as well as position appears in the response of the model. Figures 8 and 9 show the system and approximate model normalized responses for a ramp input of magnitude K with various values of T and n and unity bandwidth. Again, large n and small T give more satisfactory model response, however these values are limited in practical applications. For small T the approximate model response follows that of the system satisfactorily during the second period. For $t \geq (n + \frac{1}{2})T$ the approximate model response exhibits a constant final slope, S , where from Equation (61c)

$$S = e^{-\frac{1}{2}aT} [1 - e^{-anT}] \quad . \quad (62)$$

The system response, however, has a constantly increasing slope approaching unity as $t \rightarrow \infty$. The expression for S in Equation (62) is identical with that for V in Equation (33). Hence S has a maximum value occurring at $T = T_0$. As $T \rightarrow T_0$ from values less than T_0 , the slope S increases; however, the response $c_{1A}(t)$ follows the system response $c(t)$ less closely for $\frac{1}{2}T \leq t < (n + \frac{1}{2})T$. Again, selecting the model parameters presents a conflict between maximizing S and improving the model response for the second range of t . We choose to select n and T to maximize the final slope of the approximate model response. Thus n and T are selected the same as in the step input case.

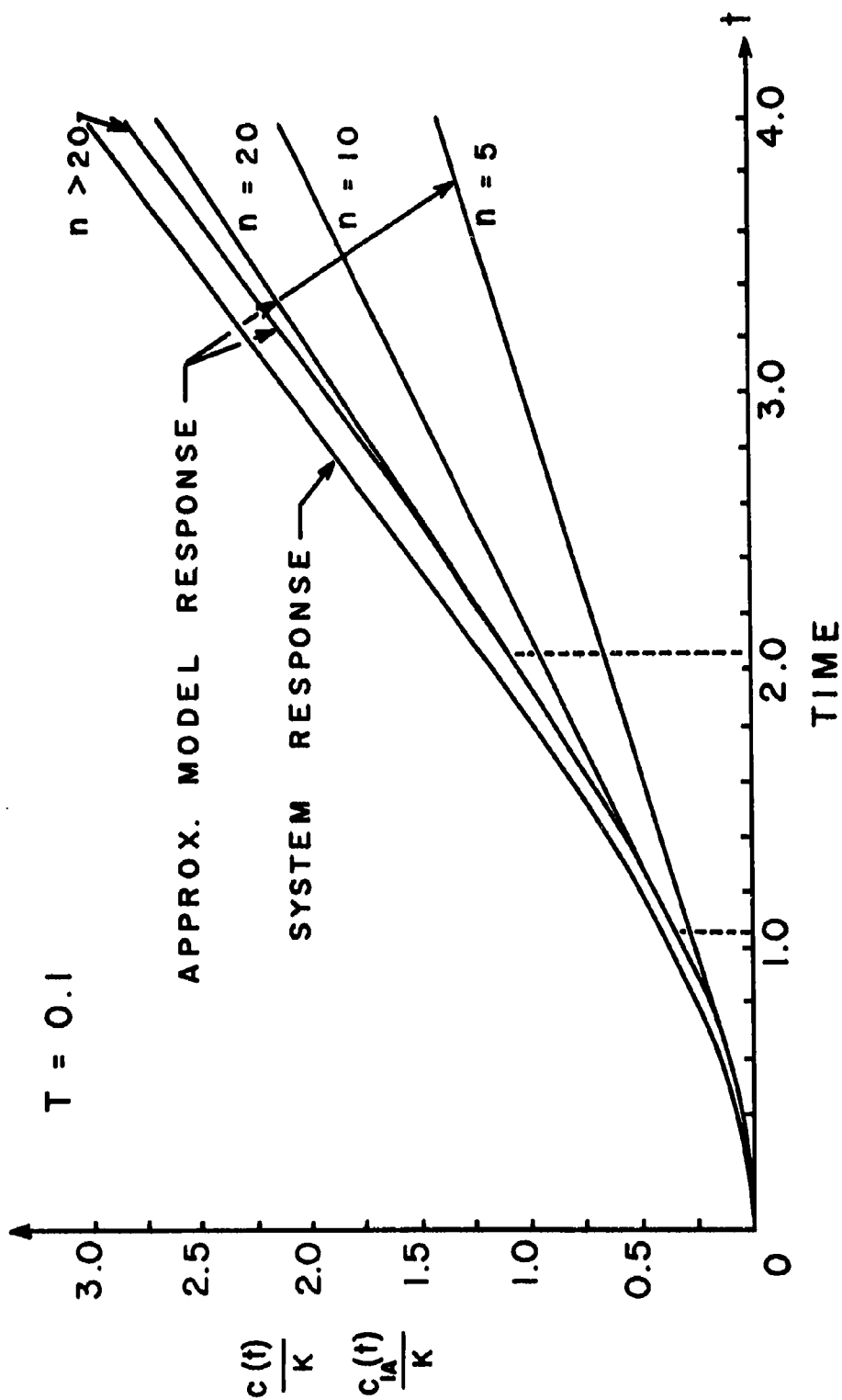


Figure 8
System and Approximate Model Responses to a
Ramp Input for $T = 0.1$

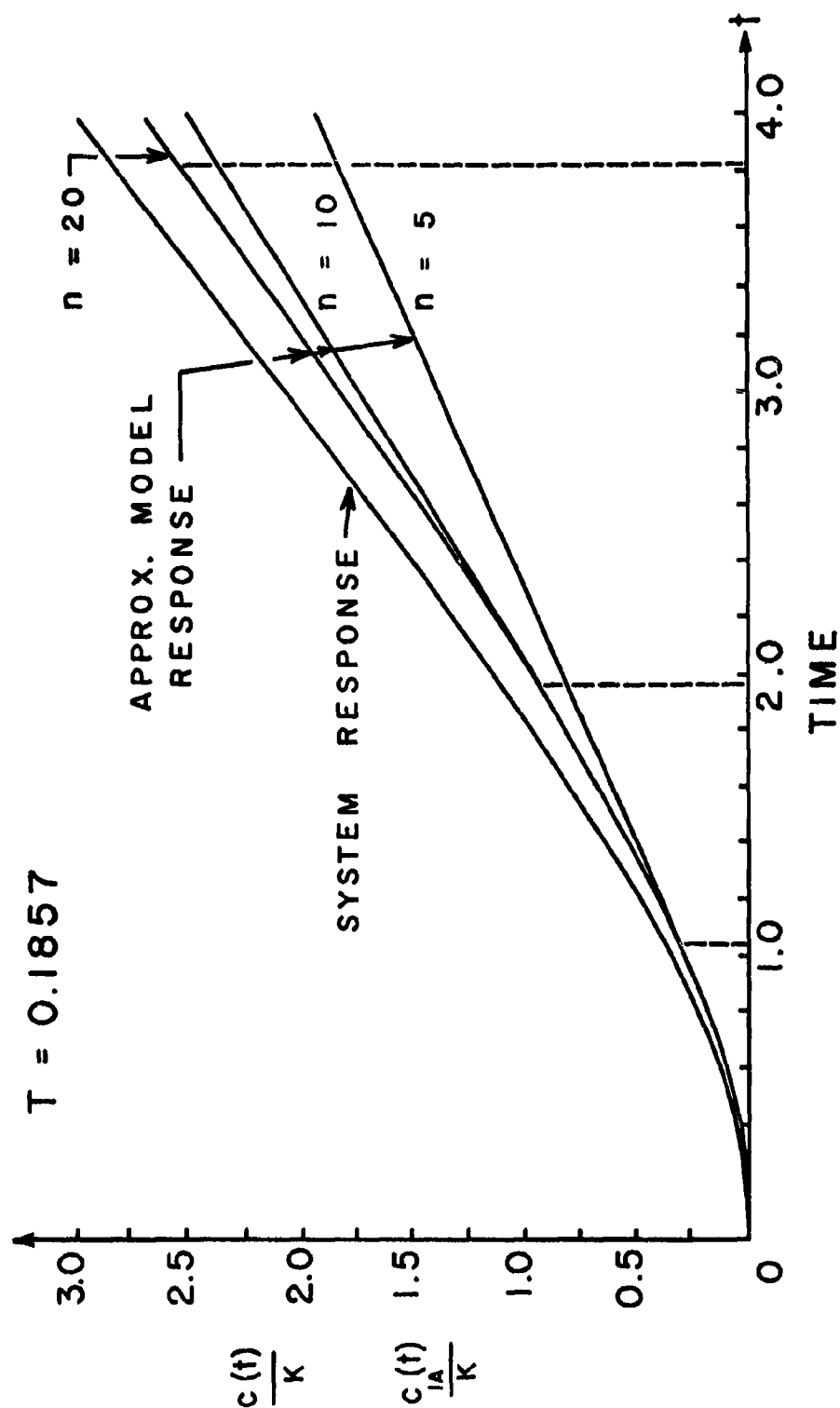


Figure 9
System and Approximate Model Responses to a Ramp
Input for $T = T_0$ at $n_0 = 20$

Comparison of the Approximate Model with the Exact Model

It is desired to determine the effects of the approximation in which Π_s in the expression for $G_1(s)$ in Equation (19) was omitted. Replacing $R(s)$ in Equation (22) by the Laplace transform of the ramp input yields

$$C_1^*(s) = \left[\frac{1}{s^2} \right] \left[\frac{a}{s + a} \right] [\Pi_s] \quad (63)$$

By expanding this expression for $C_1^*(s)$ in partial fractions and employing the inverse Laplace transform we obtain

$$c_1^*(t) = \left[\Pi_T \right] t - \frac{1}{a} + e^{-at} \left\{ \frac{1}{a} + \sum_{m=1}^{\infty} A_m \left[\sin \frac{2\Pi_m}{T} t + \theta_m \right] \right\} \quad (64)$$

where Π_T , A_m , and θ_m are given in Equations (40), (41), and (42) respectively. From Equations (24) and (64) it follows that

$$c_1(t) = 0, \quad 0 \leq t < \frac{1}{2}T \quad (65a)$$

$$c_1(t) = \left[\Pi_T \right] \left[e^{-\frac{1}{2}aT} (t - \frac{1}{2}T) \right] - \frac{1}{a} e^{-\frac{1}{2}aT} + e^{-at} \left\{ \frac{1}{a} \sum_{m=1}^{\infty} A_m \sin \left[\frac{2\Pi_m}{T} t + \theta_m - m\Pi \right] \right\}, \quad (65b)$$

$$\frac{1}{2}T \leq t < (n + \frac{1}{2})T$$

$$\begin{aligned}
c_1(t) &= \left[\prod_T \right] \left[e^{-\frac{1}{2}aT} - e^{-a(n+\frac{1}{2})T} \right] t \\
&= e^{-\frac{1}{2}aT} \left\{ \left[\prod_T \right] \left[\frac{1}{2}T - e^{-anT} (n + \frac{1}{2})T \right] + \frac{1}{a} [1 - e^{-anT}] \right\} \quad (65c) \\
&\quad t \geq (n + \frac{1}{2})T
\end{aligned}$$

Comparison of Equations (65) with Equations (61) indicates again that the effect of the approximation is the absence of the attenuating product \prod_T from the approximate model response. Also missing is the infinite sine series which causes the discontinuities in the exact model response.

To illustrate graphically the effects of the approximation, the expressions for $r(t)$ and $g(t)$ in Equations (59) and (10) are used in Equation (3), giving

$$\begin{aligned}
c_1(t) &= a t \left[e^{-aT} (t - T) u(t - T) + e^{-2aT} (t - 2T) u(t - 2T) \right. \\
&\quad \left. + \dots + e^{-naT} (t - nT) u(t - nT) \right] \quad (66)
\end{aligned}$$

Figures 10 and 11 show the system and exact model normalized responses for a ramp input of magnitude K with $a = 1$ and several values of n and T . It is recalled that the approximation for the step input resulted in the smoothing of the "staircase" exact model response. For the ramp input, however, comparison of, for example, the responses in Figure 8 with those in Figure 10, shows a smoothing of the "staircase slope" of the exact model response. A further effect of the approximation is to decrease the time at which the model response begins from T to $\frac{1}{2}T$. Also, the time at which the constant-slope portion of the model response

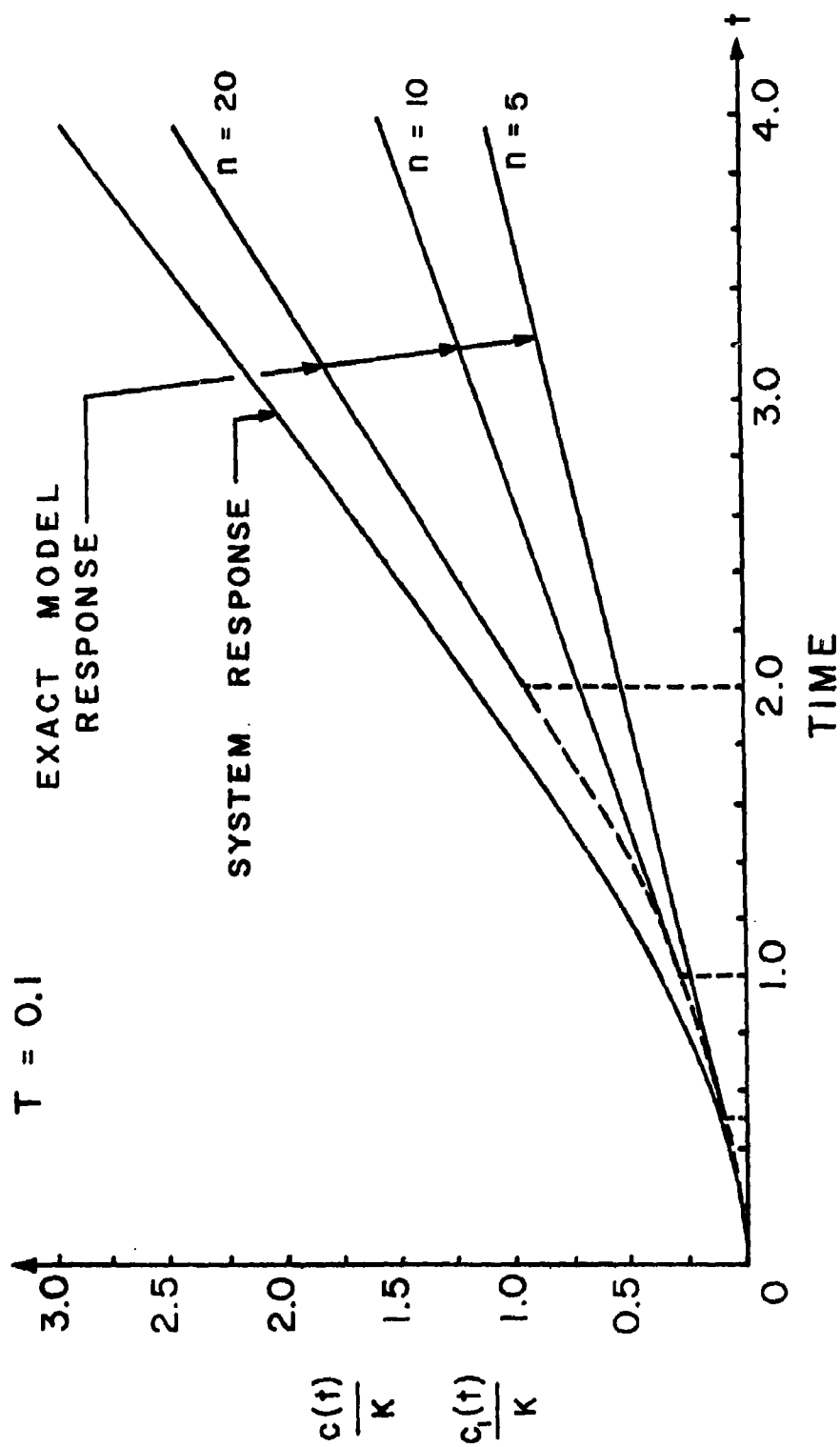


Figure 10
System and Exact Model Responses to a Ramp
Input for $T = 0.1$

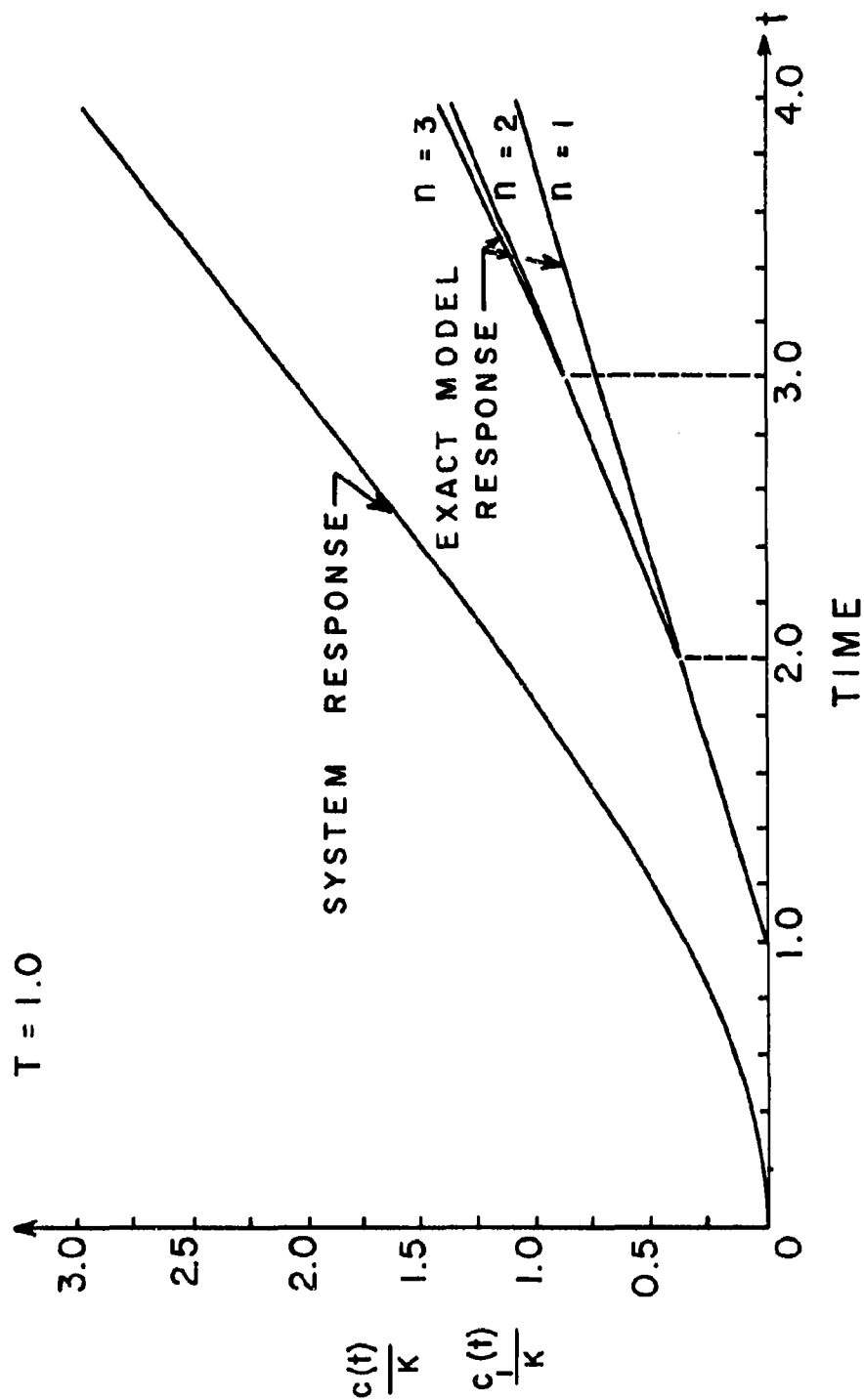


Figure 11
System and Exact Model Responses to a
Ramp Input for $T = 1.0$

begins is increased from nT to $(n + \frac{1}{2})T$. This and the smoothing are again attributed to the absence of the infinite sine series in the approximate model response.

In Figures 10 and 11 the staircase nature of the slope for the exact model response is indicated by dashes. These dashes are straight line segments and are connected. The connections are left out to make the changes in slope clear to the reader. Thus in Figure 10 each dash lasts 0.1 unit of time. One dash runs from $t = 0.1$ to $t = 0.2$, a second from $t = 0.2$ to $t = 0.3$, etc. In the case of the $n = 20$ curve, which starts at $t = 0.1$, the slope changes every 0.1 unit of time until $t = 2.0$. After $t = 2.0$ the $n = 20$ curve has a constant slope.

As in the step input study the concern is whether the results in selecting the approximate model are applicable to the exact model. We chose to first select n as large as possible for practical applications. The time T was then selected to maximize the final slope S of the approximate model response. This gave the same result for T_0 as in the step input study. From Equations (65c) and (62) the final slope, S_1 , of the exact model response is given by

$$S_1 = \left[\prod_T \right] S \quad (67)$$

Hence by Relation (58) we conclude that maximizing S rather than S_1 is a reasonable method for selecting the value of the delay time.

IMPULSE INPUT

The third input to be considered is given by

$$r(t) = \delta(t) \quad (68)$$

where the unit impulse, $\delta(t)$, is defined by

$$\left. \begin{aligned} \delta(t - t_1) &= 0, & t &\neq t_1 \\ \int_{-\epsilon + t_1}^{\epsilon + t_1} \delta(t - t_1) dt &= 1, & \epsilon &> 0 \end{aligned} \right\} \quad (69)$$

Since the input is an impulse, the weighting function in Equation (10) is used to give the system response

$$c(t) = a e^{-at} u(t) \quad (70)$$

Hence by Equation (29)

$$c_{1A}(t) = 0, \quad 0 \leq t < \frac{1}{2}T \quad (71a)$$

$$c_{1A}(t) = a e^{-at}, \quad \frac{1}{2}T \leq t < (n + \frac{1}{2})T \quad (71b)$$

$$c_{1A}(t) = 0, \quad t \geq (n + \frac{1}{2})T \quad (71c)$$

Selection of Model Parameters

For the period $\frac{1}{2} \leq t < (n + \frac{1}{2})T$, the approximate model response is identical with the response of the system. Figure 12 indicates the system and approximate model normalized responses for an impulse of strength K with $a = 1$ and various values of n . Only one value of T is used due to the simplicity of the response plots. Good model response results when the lower limit $\frac{T}{2}$ of time t in Equation (71b) is small and the upper limit $(n + \frac{1}{2})T$ is large. That is, the smaller the value of T and the larger the value of n , the better is the model response. Practical values of these parameters cause conflicts in model selection. For any T the best n is as large as possible in physical applications. On the other hand, the problem of selecting the best T for any n is that if T is chosen too small the interval

$$(n + \frac{1}{2})T - \frac{1}{2}T = nT$$

of good model response is too small. If T is selected too large, the starting time $t_s = \frac{1}{2}T$ of the model output is too great. Hence the difficulty in selecting T for the impulse study differs from that in the previous two cases. The criterion for selection is time interval and starting time of model response, or graphical abscissa, rather than values of model output or graphical ordinate. We choose T to maximize the area under the approximate model response curve as a reasonable solution to the problem. The area A_1 under this curve, found by integration, is given by

$$A_1 = e^{-\frac{1}{2}aT} [1 - e^{-anT}] \quad (72)$$

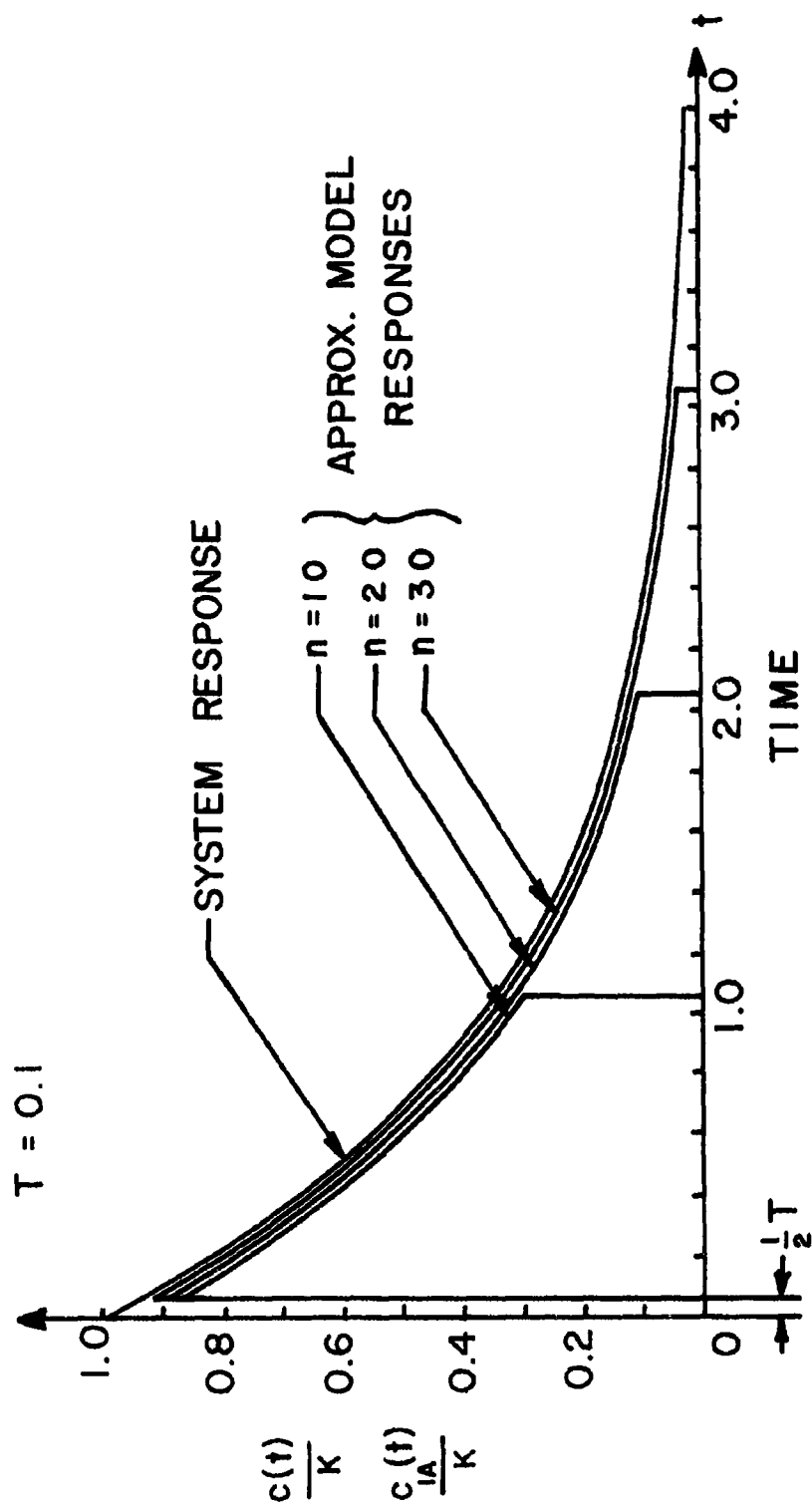


Figure 12
System and Approximate Model Responses to an
Impulse Input for $T = 0.1$

This expression is identical with that for V and S of the previous two cases. Hence the value T_0 of the delay time T to give maximum A_1 is given by Equation (34). The expression for maximum area is the same as that for V_m in Equation (35). From Equations (36) and (37), increasing n_0 lowers T_0 and increases the maximum area as before.

Comparison of the Approximate Model with the Exact Model

To determine whether using the approximation is a reasonable method of selecting n and T, the exact model response must be determined. Since $R(s) = 1$ for the impulse input, Equation (22) becomes

$$C_1^*(s) = \left[\frac{a}{s+a} \right] \left[\prod s \right] \quad . \quad (73)$$

Using partial fraction expansion and the inverse Laplace transform we obtain from Equation (73)

$$c_1^*(t) = e^{-at} \left\{ a + \sum_{m=1}^{\infty} A_m \sin \left[\frac{2\pi m}{T} t + \theta_m \right] \right\} \quad . \quad (74)$$

Thus by Equation (24)

$$c_1(t) = 0 \quad , \quad 0 \leq t < \frac{1}{2}T \quad (75a)$$

$$c_1(t) = e^{-at} \left\{ a + \sum_{m=1}^{\infty} A_m \sin \left[\frac{2\pi m}{T} t + \theta_m - m\pi \right] \right\} \quad , \quad (75b)$$

$$\frac{1}{2}T \leq t < (n + \frac{1}{2})T$$

$$c_1(t) = 0 \quad , \quad t \geq (n + \frac{1}{2})T \quad . \quad (75c)$$

Unlike the responses in the step and ramp input studies, the infinite product \prod_T does not appear in the expression for $c_1(t)$ for the impulse input. Hence the effect of omitting \prod_s from $G_1(s)$ in the simplification technique is the absence of only the infinite sine series from Equations (75).

To facilitate plotting $c_1(t)$ versus t , Equation (3) is used to give for the impulse input

$$c_1(t) = a T \left[e^{-aT} \delta(t - T) + e^{-2aT} \delta(t - 2T) \right. \\ \left. + \dots + e^{-naT} \delta(t - nT) \right] \quad (76)$$

The system and exact model normalized responses for an impulse input of weight K are shown in Figure 13, with $a = 1$, $T = 0.1$, and various values of n . The exact model response is a series of impulses corresponding to a sampling of the system response. Hence the effect of the approximation shown in Figure 12 is that of an ideal holding device over a limited time interval. Omitting the infinite sine series in the approximate model response not only changes the limits of model response from T to $\frac{1}{2}T$ and nT to $(n + \frac{1}{2})T$ but also removes the discontinuities of the exact model response.

Since the ordinates of the response $c_1(t)$ represent strengths of impulses, there is no area under the response curve to maximize in selecting T . The problem of selection, however, is similar to that in the approximation. The interval $nT - T$ is desired large, which for constant n means increasing T . Conversely, the starting time of model response and interval between the impulses is desired small. The exact model response is a pure sampling of the system response for the time interval

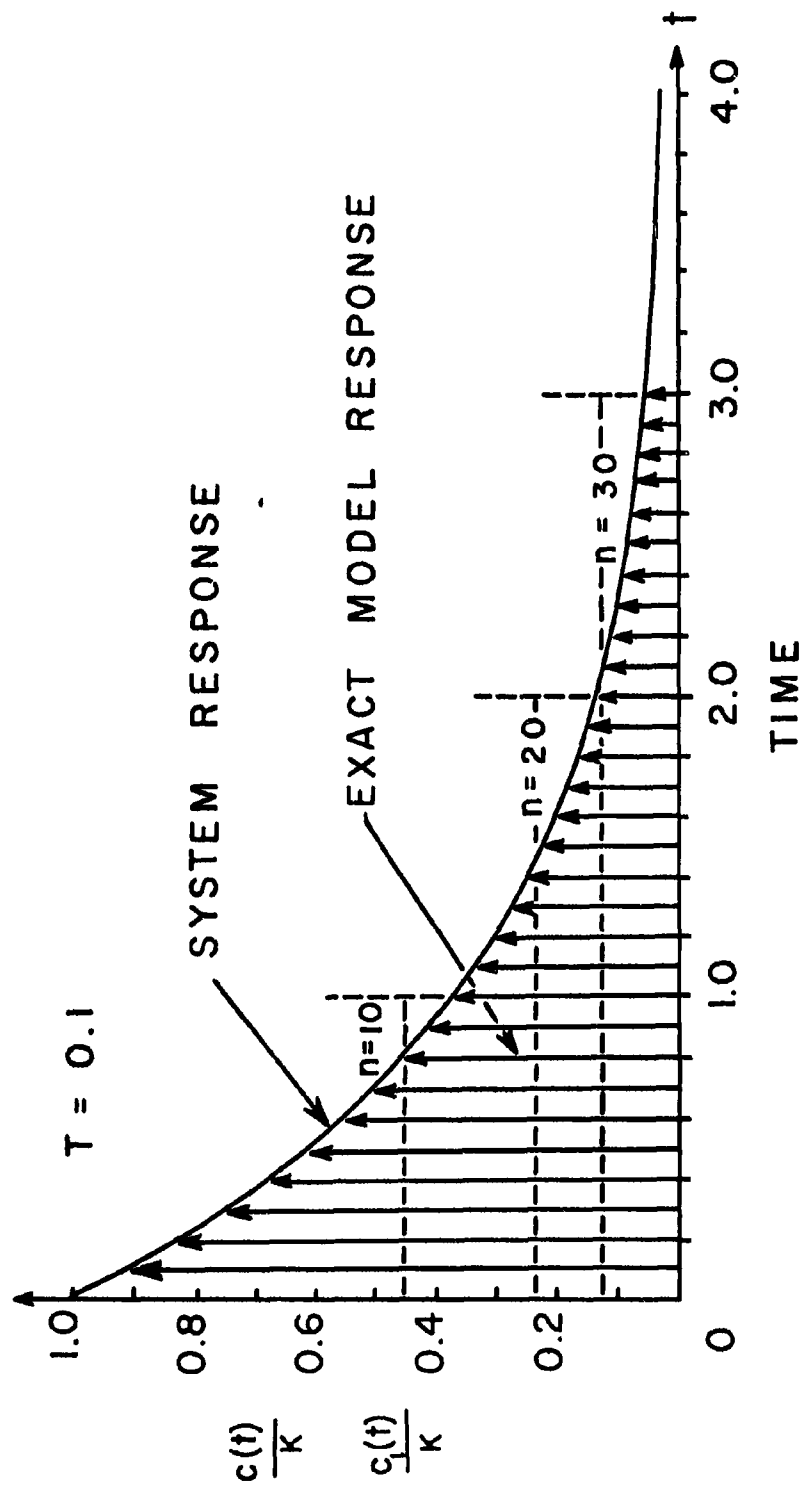


Figure 13
System and Exact Model Responses to an
Impulse Input for $T = 0.1$

of Equation (75b). Thus a technique for selecting the delay time is to maximize the area under the system response curve between limits determined by the exact model response, namely $\frac{1}{2}T \leq t < (n + \frac{1}{2})T$. But this area is exactly that given in Equation (72). Thus unlike the previous two input cases, selection of the delay time could have been effected without using the approximation.

7.

PULSE INPUT

The last transient input to be considered is the pulse shown in Figure 14a. For $K = 1$, where K is the height of the pulse, the input is given by

$$r(t) = u(t) - u(t - \tau) \quad (77)$$

where τ is the duration of the pulse. It is easily shown that the system response to this input is given by

$$c(t) = [1 - e^{-at}]u(t) - [1 - e^{-a(t-\tau)}]u(t - \tau) \quad (78)$$

From Equation (29) it follows that

$$c_{1A}(t) = 0, \quad 0 \leq t < \frac{1}{2}T \quad (79a)$$

$$\tau < nT \begin{cases} c_{1A}(t) = e^{-\frac{1}{2}aT} - e^{-at}, & \frac{1}{2}T \leq t < (\tau + \frac{1}{2}T) \quad (79b) \\ c_{1A}(t) = e^{-a(t-\tau)} - e^{-at}, & (\tau + \frac{1}{2}T) \leq t < (n + \frac{1}{2})T \quad (79c) \end{cases}$$

$$\tau \geq nT \begin{cases} c_{1A}(t) = e^{-\frac{1}{2}aT} - e^{-at}, & \frac{1}{2}T \leq t < (n + \frac{1}{2})T \quad (79d) \end{cases}$$

$$c_{1A}(t) = e^{-\frac{1}{2}aT} [1 - e^{-anT}] - [e^{-\frac{1}{2}aT} - e^{-a(t-\tau)}] u(t - \tau - \frac{1}{2}T),$$

$$(n + \frac{1}{2})T \leq t < \tau + (n + \frac{1}{2})T \quad (79e)$$

$$c_{1A}(t) = 0, \quad t \geq \tau + (n + \frac{1}{2})T. \quad (79f)$$

Selection of Model Parameters

Comparing Equations (79) with Equation (78) reveals again that the smaller the value of T and the larger the value of n , the better is the approximate response. Figure 14b shows system and approximate model normalized response for a pulse input of magnitude K with $a = 1$ and values of T , n , and τ as shown. The output of the model is equal to that of the system for $(\frac{1}{2}T + \tau) \leq t \leq (n + \frac{1}{2})T$. For t outside these limits the value of $c_{1A}(t)$ is always less than $c_1(t)$. For any T the best n is as large as practical in physical applications. If, on the other hand, n is held constant and T is reduced to improve the response of the approximate model for $\frac{1}{2}T \leq t < \tau + \frac{1}{2}T$, the response for $t \geq (n + \frac{1}{2})T$ is less desirable. If T is increased the reverse is true. As a reasonable compromise to the conflict in selecting the best delay time, pick n as large as possible. Then choose T to maximize the area under the approximate model response curve. By integrating it may be shown from Equations (79) that this area, A_p , is given by

$$A_p = \tau [1 - e^{-a\tau}] e^{-\frac{1}{2}aT} \quad (80)$$

With the exception of the pulse duration τ , the expression for A_p is the same as that for V , S , and A_1 encountered earlier. Hence the delay time to give maximum area, A_{pm} , is given by Equation (34) and

$$A_{pm} = \tau V_m \quad (81)$$

where V_m is given in Equation (35).

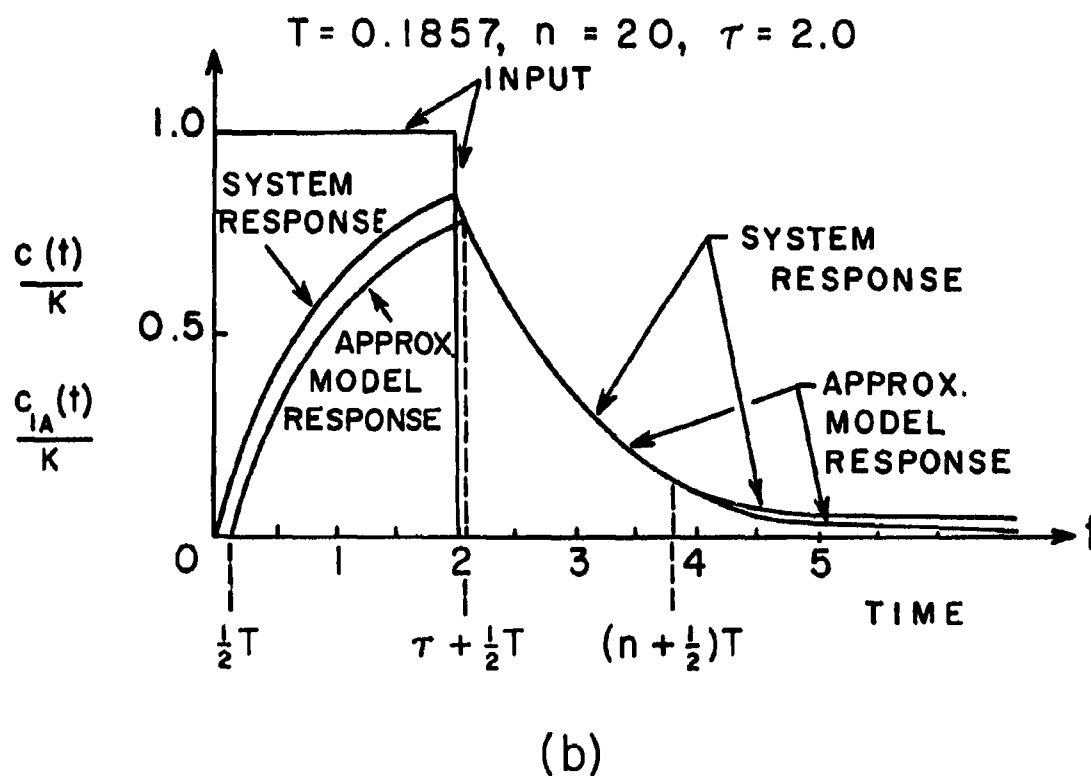
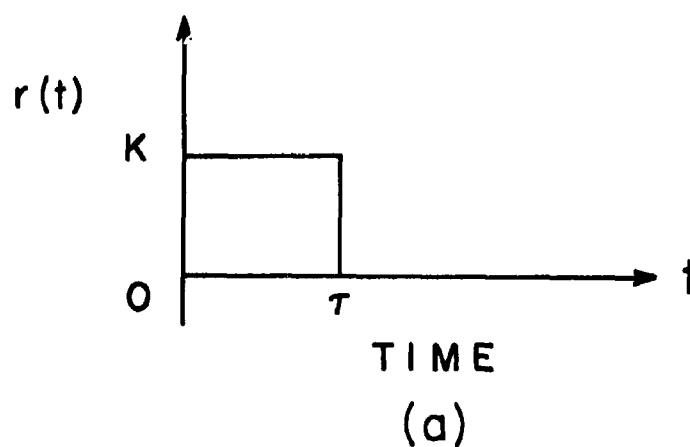


Figure 14a - Pulse Input

Figure 14b - System and Approximate Model Responses to Pulse Input

Comparison of the Approximate Model with the Exact Model

Substituting the expression for the impulse input and the system weighting function in Equation (3) we obtain

$$\begin{aligned} c_1(t) = & a T \left\{ e^{-aT} [u(t - T) - u(t - T - \tau)] u(t - T) \right. \\ & + e^{-2aT} [u(t - 2T) - u(t - 2T - \tau)] u(t - 2T) \quad (82) \\ & + \dots + e^{-naT} [u(t - nT) - u(t - nT - \tau)] u(t - nT) \left. \right\}. \end{aligned}$$

By inspection, the area A_1 under the curve given by a plot of $c_1(t)$ is given by

$$A_1 = \tau [a T] [e^{-aT} + e^{-2aT} + \dots + e^{-naT}]. \quad (83)$$

But the product of the bracketed quantities in this expression is easily seen to be the final value of $c_1(t)$ in Equation (48). From Equation (49) it follows that

$$A_1 = \tau \left[\frac{1}{T} \right] [1 - e^{-anT}] e^{-\frac{1}{2}aT}. \quad (84)$$

By Relation (58) it is concluded that choosing T to maximize A_p in Equation (80) rather than A_1 is a reasonable technique for selecting the delay time.

8.

SINUSOIDAL INPUT

Consider lastly the sinusoidal input given by

$$r(t) = \sin(\omega t) u(t) \quad (85)$$

where ω is the signal frequency. Let the quantities Q_1 , Q_2 , and φ be defined by

$$Q_1 = \frac{a\omega}{a^2 + \omega^2} \quad , \quad (86)$$

$$Q_2 = \frac{a}{(a^2 + \omega^2)^{\frac{1}{2}}} \quad , \quad (87)$$

$$\varphi = \tan^{-1} \left(\frac{\omega}{a} \right) \quad . \quad (88)$$

Then for the approximation study we follow procedures similar to those in the previous four cases to obtain

$$c_{1A}(t) = 0 \quad , \quad 0 \leq t < \frac{1}{2}T \quad (89a)$$

$$c_{1A}(t) = Q_1 e^{-at} + Q_2 e^{-\frac{1}{2}aT} \sin \left[\omega t - \varphi - \frac{1}{2}\omega T \right] \quad , \quad (89b)$$

$$\frac{1}{2}T \leq t < (n + \frac{1}{2})T$$

$$c_{1A}(t) = Q_2 e^{-\frac{1}{2}aT} \left[\sin(\omega t - \varphi - \frac{1}{2}\omega T) - e^{-anT} \sin(\omega t - \varphi - \frac{1}{2}\omega T - \omega n T) \right], \quad (89c)$$

$$t \geq (n + \frac{1}{2})T.$$

Selection of Model Parameters

The response of the system to the sinusoidal input is given by

$$c(t) = \left\{ Q_1 e^{-at} + Q_2 \sin[\omega t - \varphi] \right\} u(t). \quad (90)$$

From Equation (89b) decreasing the value of T improves the model response during the specified interval by increasing the value of $e^{-\frac{1}{2}aT}$ toward one and decreasing the value of the phase lag $\frac{1}{2}\omega T$ toward zero. On the other hand, from Equations (89b) and (89c), too small a value for T gives poor over-all model response. For any T the number n should be chosen as large as possible. Again some compromise is needed in selecting T . We compare $c_{1A}(t)$ with $c_1(t)$ for $t \geq (n + \frac{1}{2})T$. Assume that the quantity $(n + \frac{1}{2})T$ is large enough such that the exponential in Equation (90) may be neglected, giving

$$c(t) \approx Q_2 \sin[\omega t - \varphi], \quad t \gg \frac{1}{a}. \quad (91)$$

The first sine term of Equation (89c) contains a phase lag of $\frac{1}{2}\omega T$ radians from the sine term of the system response. The second sinusoid lags the first by $\omega n T$ radians. However, assuming that

$$e^{-\frac{1}{2}aT} \gg e^{-a(n+\frac{1}{2})T} \quad (92)$$

the effect on $c_{1A}(t)$ of the second sinusoid of Equation (89c) will be

negligible regardless of phase shift $\omega n T$. Conversely, if the second sine term with its attenuating exponential is neglected completely the parameter n will be absent from the remaining terms. No method for selecting T would then be available. Hence as a compromise we write

$$c_{1A}(t) \approx Q_2 e^{-\frac{1}{2}aT} [1 - e^{-anT}] \sin \left[\omega t - \varphi - \frac{1}{2}\omega T \right], \quad (93)$$

$$t \geq (n + \frac{1}{2})T; \quad (n + \frac{1}{2})T \gg \frac{1}{a}; \quad nT \gg \frac{1}{2}T$$

in which the effect of the phase shift $\omega n T$ is neglected. Comparison of Relation (93) with Relation (91) indicates that T should be chosen to minimize the shift $\frac{1}{2}\omega T$ and maximize the quantity $e^{-\frac{1}{2}aT} [1 - e^{-anT}]$. However, since the system is linear, the delay time should not be a function of the signal frequency ω . Hence, we select T only to maximize the attenuating expression above. The input frequency must be kept small in order that the model response phase lag be small. Thus the model parameters n and T are selected as in the previous four cases, the delay time T_0 being given by Equation (34). Figure 15 shows the system and approximate model normalized responses as given by Relations (91) and (93) for a sinusoidal input of magnitude K . The values of the parameters used are $a = 1$, $\omega = 1$, $n = 20$, and $T = T_0$.

Comparison of the Approximate Model with the Exact Model

To determine the exact model response we replace $R(s)$ in Equation (22) by the Laplace transform of the sinusoidal input to obtain

$$C_1^*(s) = \left[\frac{\omega}{s^2 + \omega^2} \right] \left[\frac{a}{s + a} \right] \left[\frac{1}{s} \right] \quad (94)$$

$T = 0.1857, \quad n = 20, \quad \omega = 1.0$

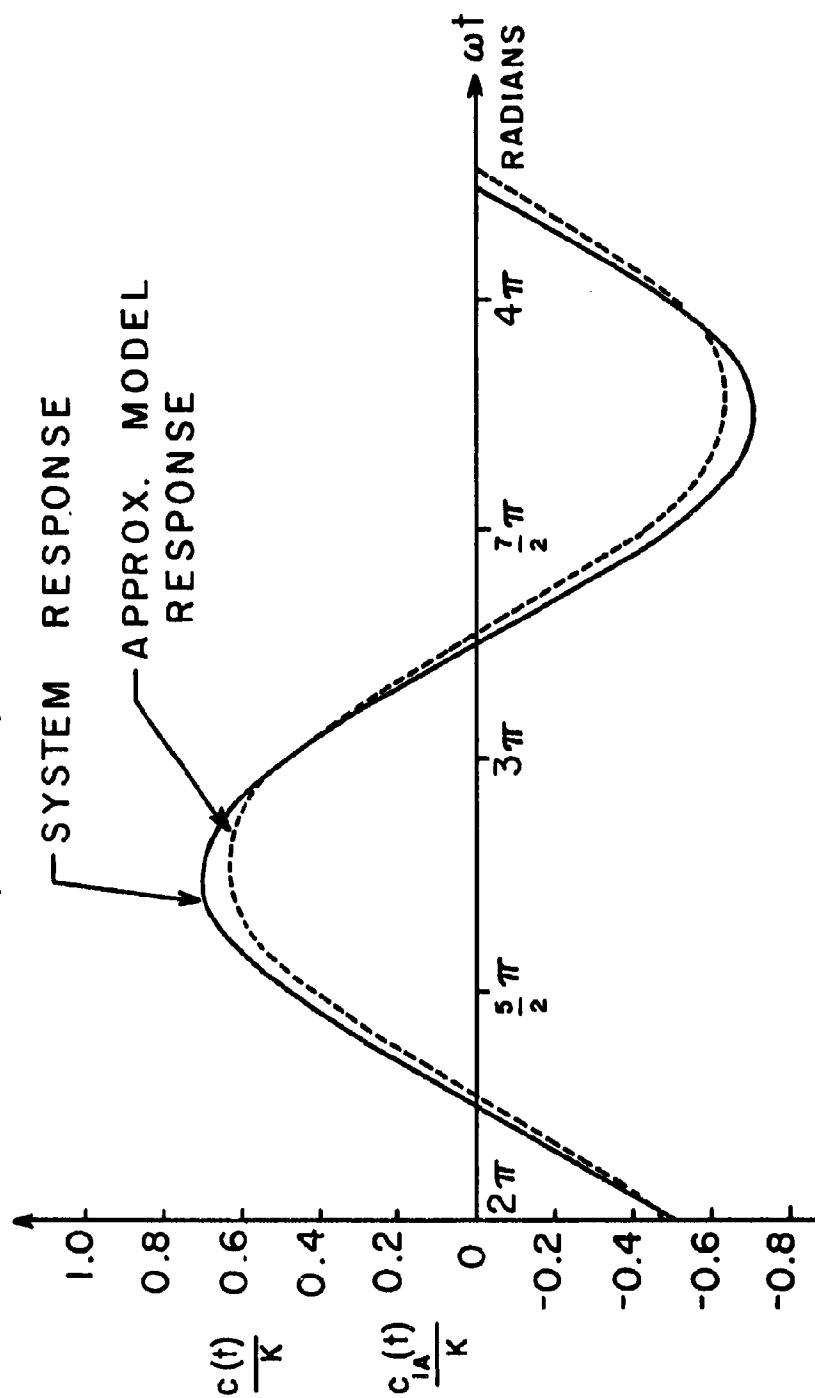


Figure 15

System and Approximate Model Responses Given by Relations 91 and 93 to a Sinusoidal Input

Let the quantity $|B_m|$ be the absolute value of B_m , where

$$B_m = \frac{a}{a + j\omega} \prod_{m=1}^{\infty} \left\{ \frac{1}{1 + \frac{\tau^2(a + j\omega)^2}{4 \pi^2 m^2}} \right\} \quad (95)$$

and $j = \sqrt{-1}$ as before. We define the quantity $\angle B_m$ to be the arc-tangent of the quotient obtained by dividing the imaginary part of B_m by the real part. Deriving the expression for $c_1^*(t)$ by the inverse Laplace transform, we have from Equation (24)

$$c_1(t) = 0, \quad 0 \leq t < \frac{1}{2}T \quad (96a)$$

$$c_1(t) = e^{-at} \left\{ C_1 + \sum_{m=1}^{\infty} A_m \sin \left[\frac{2\pi m}{T} t + \theta_m - m\pi \right] \right\} \\ + |B_m| e^{-\frac{1}{2}aT} \sin \left[\omega t + \angle B_m - \frac{1}{2} \omega T \right], \quad (96b)$$

$$\frac{1}{2}T \leq t < (n + \frac{1}{2})T$$

$$c_1(t) = |B_m| e^{-\frac{1}{2}aT} \left\{ \sin \left[\omega t + \angle B_m - \frac{1}{2} \omega T \right] \right. \\ \left. - e^{-anT} \sin \left[\omega t + \angle B_m - \frac{1}{2} \omega T - \omega n T \right] \right\}, \quad (96c)$$

$$t \geq (n + \frac{1}{2})T.$$

Neglecting the infinite series in Equation (96b), we obtain Equations (89) from Equations (96) by replacing $|B_m|$ and $\angle B_m$ with Q_2 and $-\varphi$ respectively. Since the model parameters were selected using Relation (93), we must determine whether the approximations

$$-\varphi \approx \angle B_m \quad (97)$$

$$Q_2 \approx |B_m| \quad (98)$$

are reasonable.

Introduce σ_m , where

$$\sigma_m = \sum_{m=1}^{\infty} \left\{ \tan^{-1} \left[\frac{2a \omega T^2}{4\pi^2 m^2 + T^2(a^2 - \omega^2)} \right] \right\} . \quad (99)$$

For the approximation of Relation (97) it may be easily shown from Equation (95) that

$$\angle B_m = -\varphi - \sigma_m \quad (100)$$

where φ is given in Equation (88). Let k_ω be a non-negative number such that

$$\omega = k_\omega a \quad (101)$$

Then it is shown in Appendix A that

$$0 \leq \sigma_m < \frac{1}{48} , \quad k_\omega \leq 1 \quad (102)$$

$$0 \leq \sigma_m < \pi , \quad k_\omega > 1 . \quad (103)$$

Consequently, if the value of ω is greater than the bandwidth of the system, it is necessary to establish a maximum for k_ω in order that the approximation of Relation (97) be reasonable. By reasoning similar to that preceding Relation (93) we have from Equation (96c)

$$c_1(t) \approx |B_m| e^{-\frac{1}{2}aT} \left[1 - e^{-anT} \right] \sin \left[\omega t + \angle B_m - \frac{1}{2} \omega T \right], \quad (104)$$

$$t \geq (n + \frac{1}{2})T; \quad (n + \frac{1}{2})T \gg \frac{1}{a}; \quad nT \gg \frac{1}{2}T.$$

Let a new quantity ψ_m be designated the exact model phase lag, where from Relations (91) and (104)

$$\psi_m = -\varphi - \left[\angle B_m - \frac{1}{2} \omega T \right]. \quad (105)$$

Substituting the expression for φ and $\angle B_m$ yields

$$\psi_m = \frac{1}{2} \omega T + \sigma_m. \quad (106)$$

By the restrictions on the product aT in Relation (50) we obtain from Relations (102) and (103)

$$0 \leq \psi_m < \frac{1 + 12k\omega}{48}, \quad k\omega \leq 1 \quad (107)$$

$$0 \leq \psi_m < \frac{k\omega}{4} + \pi, \quad k\omega > 1. \quad (108)$$

Hence for $\omega \leq a$ we are assured that ψ_m can be no larger than 0.271 radians, or approximately 15 degrees. By Relation (108) this constraint on the exact model phase lag increases for $\omega > a$. We intuitively assume that a phase lag greater than 15 degrees would not meet nominal engineering requirements. Thus ω is restricted to values less than or equal to the system bandwidth. The approximation of $\angle B_m$ by $-\varphi$ is then reasonable from Relations (100) and (102).

To justify Relation (98) introduce Π_ω , where

$$\Pi_{\omega} = \prod_{m=1}^{\infty} \left[\left\{ \left[1 + \frac{T^2(a^2 - \omega^2)}{4\pi^2 m^2} \right]^2 + \left[\frac{2a\omega T^2}{4\pi^2 m^2} \right]^2 \right\}^{\frac{1}{2}} \right] \quad (109)$$

From Equation (95) it follows that

$$|B_m| = Q_2 \left[\frac{1}{\Pi_{\omega}} \right] \quad (110)$$

Let a new term γ be defined by

$$\gamma = \frac{a^2 T^2}{4\pi^2} \quad (111)$$

Then by Relation (50)

$$\gamma < \frac{1}{16\pi^2} \quad (112)$$

It is shown in Appendix B that

$$\exp \left\{ \frac{k_{\omega}^2 - 2k_{\omega} - 1}{96} \right\} \leq \frac{1}{\Pi_{\omega}} \leq \left\{ \left[\frac{|k_{\omega}^2 - 1|}{\frac{1}{\gamma} - |k_{\omega}^2 - 1|} \right] \left[\frac{\pi^2}{6} \right] \right\}, \quad (113)$$

$$0 \leq k_{\omega} < \left[1 + \frac{1}{\gamma} \right]^{\frac{1}{2}}.$$

By Relation (112)

$$\left[1 + \frac{1}{\gamma} \right]^{\frac{1}{2}} > 4\pi \quad (114)$$

Since k_{ω} has already been restricted to values less than or equal to one, we have from Relation (113)

$$0.979 \leq \frac{1}{\pi_{\omega}} \leq 1.011, \quad 0 \leq k_{\omega} \leq 1. \quad (115)$$

From Equation (110) we conclude that the approximation in Relation (98) is reasonable.

9.

SECOND ORDER SYSTEM

To note some of the difficulties in selecting the delay line model for more complex systems, consider the transfer function of a second order system

$$G(s) = \frac{b c}{(s + b)(s + c)}, \quad b \neq c \quad (116)$$

where b and c are either both real non-zero numbers or are non-zero complex conjugates. Following a procedure similar to that for Equations (9) through (19) we obtain

$$G_1(s) = \left\{ \frac{b c}{c - b} \left\{ \frac{e^{-\frac{1}{2}bT}e^{-\frac{1}{2}Ts} - e^{-(n+\frac{1}{2})bT}e^{-(n+\frac{1}{2})Ts}}{(b + s) \prod_{m=1}^{\infty} \left[1 + \frac{T^2(b + s)^2}{4 \Pi^2 m^2} \right]} \right. \right. \\ \left. \left. - \frac{e^{-\frac{1}{2}cT}e^{-\frac{1}{2}Ts} - e^{-(n+\frac{1}{2})cT}e^{-(n+\frac{1}{2})Ts}}{(c + s) \prod_{m=1}^{\infty} \left[1 + \frac{T^2(c + s)^2}{4 \Pi^2 m^2} \right]} \right\} \right. \quad (117)$$

Let the quantities $g_b^*[t]$ and $g_c^*[t]$ be given by

$$g_b^*[t] = \frac{b c}{b(c - b)} [1 - e^{-bt}] \quad (118)$$

$$g_c^*[t] = \frac{b c}{c(c-b)} [1 - e^{-bt}] \quad (119)$$

By omitting the infinite products in Equation (117) it may be shown that the approximate model response to a unit step input becomes

$$\begin{aligned} c_{1A}(t) &= e^{-\frac{1}{2}bT} g_b^*[t - \frac{1}{2}T] u(t - \frac{1}{2}T) \\ &- e^{-(n+\frac{1}{2})bT} g_b^*[t - (n + \frac{1}{2})T] u[t - (n + \frac{1}{2})T] \\ &- e^{-\frac{1}{2}cT} g_c^*[t - \frac{1}{2}T] u(t - \frac{1}{2}T) \\ &+ e^{-(n+\frac{1}{2})cT} g_c^*[t - (n + \frac{1}{2})T] u[t - (n + \frac{1}{2})T] \end{aligned} \quad (120)$$

The final value, V , of the approximate model response is given by

$$V = \frac{b c}{c-b} \left\{ \frac{1}{b} e^{-\frac{1}{2}bT} [1 - e^{-bnT}] - \frac{1}{c} e^{-\frac{1}{2}cT} [1 - e^{-cnT}] \right\} \quad (121)$$

The problem of selecting T to maximize V is more complicated than for the first order system. Taking the derivative of the expression for V with respect to T , equating the result to zero, and rearranging, we obtain

$$\frac{e^{-\frac{1}{2}bT}}{e^{-\frac{1}{2}cT}} = \frac{1 - (2n+1)e^{-cnT}}{1 - (2n+1)e^{-bnT}} \quad (122)$$

An explicit expression for T_0 in terms of n , c , and b cannot in general be found from Equation (122). If the input to the system is a ramp, the final slope S of the approximate model response is also given by the expression for V . Further study is therefore necessary to find a method for selecting the model parameters.

10.

SUMMARY AND CONCLUSIONS

A delay line model of a linear system is selected such that the responses of the model to commonly occurring inputs are closest to corresponding responses of the system. It is found for each input that the number n of model delay elements should be chosen as large as practical for applications. The selected value, T_0 , of the delay time, T , is then given by

$$T_0 = \frac{\ln(2n_0 + 1)}{a n_0}$$

where a is the system bandwidth and n_0 the selected n . For the periodic input studied the signal frequency is limited to values equal to or less than the system bandwidth in order that the model be acceptable.

The expression for T_0 is obtained by a technique using infinite products previously employed successfully by Oldenburger and Goodson⁶ in distributed parameter studies. By assuming a small value of the model delay time the infinite product appearing in the model transfer function is neglected and a simplified model representation obtained. It is shown that this approximation is reasonable for the transient inputs if $aT < \frac{1}{2}$. The same is true for the periodic input if in addition the input frequency is less than or equal to the system bandwidth. Although the approximation method is quite useful in the first-

order system study, new difficulties noted at the conclusion of the investigation arise for more complex systems. Since T_0 is a function of the system parameters as well as the selected number of delay elements, the difficulty in model selection increases as the number of system parameters increases.

LIST OF REFERENCES

1. Truxal, J. G., "Identification of Process Dynamics," Adaptive Control Systems, edited by E. Miskin and L. Braun, Jr., McGraw-Hill Book Company, Inc., New York, 1961, pp. 71 - 87.
2. Goodman, T. P., and Reswick, J. B., "Determination of System Characteristics from Normal Operating Records," Transactions of the American Society of Mechanical Engineers, Vol. 78, No. 2, 1956, pp. 259 - 271.
3. Tustin, A., "A Method of Analyzing the Behavior of Linear Systems in Terms of Time Series," Journal of the Institute of Electrical Engineers, Vol. 94, No. 1, Part II-A, 1947, p. 130.
4. Lewis, N. W., "Waveform Computation by the Time Series Method," Proceedings of the Institution of Electrical Engineers, Vol. 99, No. 61, Part 3, 1952.
5. Chang, C. M., "A New Technique of Determining System Characteristics from Normal Operating Records," Mechanical Engineer's Thesis, Massachusetts Institute of Technology, January, 1955.
6. Oldenburger, R., and Goodson, R. E., "Simplification of Hydraulic Line Dynamics by Use of Infinite Products," Transactions of the

American Society of Mechanical Engineers, Paper No. 62-WA-55.

7. Truxal, J. G., Automatic Feedback Control System Synthesis, McGraw-Hill Book Company, Inc., New York, 1955, pp. 55 - 56.
8. LePage, W. R., Complex Variables and the Laplace Transform for Engineers, McGraw-Hill Book Company, Inc., New York, 1961, pp. 342 - 343, 310.
9. Whittaker, E. T., and Watson, G. N., A Course of Modern Analysis, Fourth Edition, Cambridge University Press, Cambridge, Massachusetts, 1940, pp. 136 - 137.
10. Tou, J. T., Digital and Sampled-Data Control Systems, McGraw-Hill Book Company, Inc., New York, 1959, p. 80.
11. C. R. C. Standard Mathematical Tables, Twelfth Edition, Chemical Rubber Publishing Company, Cleveland, Ohio, 1959, p. 376.
12. Ferrar, W. L., A Textbook of Convergence, Oxford University Press, Oxford, England, 1938, pp. 146 - 147.
13. Knopp, K., Theory and Application of Infinite Series, Blackie and Son, London, 1928, p. 375.

APPENDIX A

ESTABLISHMENT OF CONSTRAINTS ON σ_m

Let the quantity x_m be given by

$$x_m = \frac{2a \omega T^2}{4\pi^2 m^2 + T^2(a^2 - \omega^2)} \quad (123)$$

Then from Equation (99)

$$\sigma_m = \sum_{m=1}^{\infty} \tan^{-1}(x_m) \quad (124)$$

From Equation (95) it is easily shown that

$$0 \leq \sigma_m < \pi, \quad \omega \geq 0 \quad (125)$$

for finite ω . Hereafter we consider only finite, non-negative values of ω . By Equation (123) it follows that

$$-1 \leq x_m \leq 0, \quad 1 \leq m \leq \frac{T}{2\pi} \left[\omega^2 - 2a\omega - a^2 \right]^{\frac{1}{2}} \quad (126)$$

$$-\infty \leq x_m < -1, \quad \frac{T}{2\pi} \left[\omega^2 - 2a\omega - a^2 \right]^{\frac{1}{2}} < m \leq \frac{T}{2\pi} \left[\omega^2 - a^2 \right]^{\frac{1}{2}} \quad (127)$$

$$1 < x_m < \infty, \quad \frac{T}{2\pi} \left[\omega^2 - a^2 \right]^{\frac{1}{2}} < m < \frac{T}{2\pi} \left[\omega^2 + 2a\omega - a^2 \right]^{\frac{1}{2}} \quad (128)$$

$$0 \leq x_m \leq 1, \quad m \geq \frac{\pi}{2\pi} [\omega^2 + 2a\omega - a^2]^{\frac{1}{2}}. \quad (129)$$

We recall that m in the above equations is a positive integer. Let m_1 be the first positive integer (FPI) less than or equal to, depending on the value of ω , the upper limit on m in Relation (126). From Relation (127) let m_2 be the FPI greater than this expression. The term m_3 is the FPI less than or equal to the upper limit on m in Relation (128), and so on with the next two relations through m_6 . With the series representations

$$\tan^{-1} x = \pi + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots, \quad x^2 \leq 1 \quad (130)$$

$$\frac{3\pi}{4} \leq \tan^{-1} x \leq \frac{5\pi}{4}$$

$$\tan^{-1} x = \frac{1}{2}\pi - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots, \quad x^2 \geq 1 \quad (131)$$

$$\frac{\pi}{4} \leq \tan^{-1} x \leq \frac{3\pi}{4}$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots, \quad x^2 \leq 1 \quad (132)$$

$$-\frac{\pi}{4} \leq \tan^{-1} x \leq \frac{\pi}{4}$$

we keep only part of each series to give from Equations (126) through (129)

$$0 \leq \sigma_m < \left\{ \sum_{m=1}^{m_1} \left[\pi + x_m - \frac{1}{3} x_m^3 \right] + \sum_{m=m_2}^{m_3} \left[\frac{1}{2} \pi - \frac{1}{x_m} \right] \right. \\ \left. + \sum_{m=m_4}^{m_5} \left[\frac{1}{2} \pi - \frac{1}{x_m} + \frac{1}{3x_m^3} \right] + \sum_{m=m_6}^{\infty} [x_m] \right\}. \quad (133)$$

If $\omega \leq a$, then m is defined only for $m \geq m_6$. For instance, in Relation (129) let $\omega = k\omega a$, where $0 \leq k\omega \leq 1$. Then since $aT < \frac{1}{2}$

$$\frac{T}{2\pi} [\omega^2 + 2a\omega - a^2]^{\frac{1}{2}} < \frac{\sqrt{2}}{4\pi}.$$

Hence from Relation (133)

$$0 \leq \sigma_m < \sum_{m=m_6}^{\infty} [x_m], \quad \omega \leq a. \quad (134)$$

Let quantities P and Q be given by

$$P = \frac{2a\omega T^2}{2\pi^2} \quad (135)$$

$$Q = \frac{T^2(a^2 - \omega^2)}{4\pi^2} \quad (136)$$

Then from Equation (123)

$$x_m = P \left[\frac{1}{m^2 + Q} \right] \quad (137)$$

It follows that

$$x_m < P \left[\frac{1}{m^2} \right] , \quad \omega \leq a . \quad (138)$$

By Equation (55) we obtain

$$P \sum_{m=m_0}^{\infty} \left[\frac{1}{m^2} \right] \leq \frac{a \omega T^2}{12} , \quad \omega \leq a . \quad (139)$$

Therefore from Relation (50)

$$0 \leq \sigma_m < \frac{1}{48} , \quad \omega \leq a \quad (140)$$

and Relation (102) is established.

For $\omega > a$ we have by Relation (125)

$$\sigma_m < \Pi , \quad k_{\omega} > 1 \quad (141)$$

This establishes Relation (103).

APPENDIX B

ESTABLISHMENT OF CONSTRAINTS ON Π_ω

Employing Equations (101) and (111) we obtain from Equation (109)

$$\Pi_\omega = \prod_{m=1}^{\infty} \left[\left\{ \left[1 + \gamma \frac{(1 - k_\omega^2)}{m^2} \right]^2 + \left[2\gamma \frac{k_\omega}{m^2} \right]^2 \right\}^{\frac{1}{2}} \right] . \quad (142)$$

It follows that

$$\Pi_\omega \leq \prod_{m=1}^{\infty} \left[\left| 1 + \gamma \frac{(1 - k_\omega^2)}{m^2} \right| + \left| 2\gamma \frac{k_\omega}{m^2} \right| \right] . \quad (143)$$

Let k_1 be defined by

$$k_1 = \left[\frac{\gamma + 1}{\gamma} \right]^{\frac{1}{2}} . \quad (144)$$

Then for the quantity in the first pair of vertical bars in Relation (143),

$$\left[1 + \gamma \frac{(1 - k_\omega^2)}{m^2} \right] > 0 , \quad 0 \leq k_\omega < k_1 . \quad (145)$$

For the remainder of this work the limits on k_ω in Equation (145) are assumed.

Introduce Q_m , where

$$Q_m = \frac{\gamma}{m^2} [k_\omega^2 - 2k_\omega - 1] \quad (146)$$

Whence, by Relation (145),

$$\prod_{m=1}^{\infty} [1 - Q_m] \quad (147)$$

From Equation (146) and Relation (112),

$$-\frac{1}{8\pi^2} < Q_m < 1 \quad (148)$$

for any m . Let the quantity u_m be given by

$$u_m = \sum_{n=1}^{\infty} \ln [1 - Q_m] \quad (149)$$

Then

$$\prod_{m=1}^{\infty} [1 - Q_m] = e^{u_m} \quad (150)$$

The following established relationships are noted:¹²

$$\ln (1 + x) \leq x, \quad x \geq 0 \quad (151)$$

$$\frac{y}{y-1} \leq \ln (1 - y) \leq -y, \quad 0 \leq y < 1 \quad (152)$$

Thus from Equation (149)

$$u_m \leq - \sum_{m=1}^{\infty} Q_m \quad (153)$$

Substituting the expression for Q_m in this relation and employing Equation (55) gives

$$u_m \leq \frac{-k_{\omega}^2 + 2k_{\omega} + 1}{96} \quad (154)$$

Hence

$$\prod_{\omega} \leq \exp \left[\frac{-k_{\omega}^2 + 2k_{\omega} + 1}{96} \right], \quad 0 \leq k_{\omega} < k_1 \quad (155)$$

In order to establish a lower constant for \prod_{ω} , introduce R_m , where

$$R_m = \frac{\gamma}{m^2} (k_{\omega}^2 - 1) \quad (156)$$

From Equation (142) it follows that

$$\prod_{\omega} \geq \prod_{m=1}^{\infty} [1 - R_m] \quad (157)$$

From Relation (112) the limits on R_m are given by

$$-\frac{1}{16\pi^2} \leq R_m < 1 \quad (158)$$

Let v_m be defined by

$$v_m = \sum_{n=1}^{\infty} \text{Ln} [1 - R_m] \quad . \quad (159)$$

Hence

$$\prod_{m=1}^{\infty} [1 - R_m] = e^{v_m} \quad . \quad (160)$$

Define the quantity R by

$$R = \gamma |k_0^2 - 1| \quad . \quad (161)$$

Thus

$$\frac{|R_m|}{|R_m| - 1} = \frac{R}{R - m^2} \quad . \quad (162)$$

Since from Relation (112)

$$\gamma |k_0^2 - 1| < m^2 \quad (163)$$

for all m , we have

$$\frac{R}{R - m^2} \geq \left[\frac{R}{R - 1} \right] \frac{1}{m^2} \quad . \quad (164)$$

We note that¹²

$$\text{Ln} (1 + |R_m|) \geq \text{Ln} (1 - |R_m|) \geq \frac{|R_m|}{|R_m| - 1} \quad , \quad (165)$$

$$0 \leq |R_m| < 1 \quad .$$

Thus from Equation (159) and Relations (164) and (165),

$$v_m \geq \left[\frac{R}{R-1} \right] \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \right] \quad (166)$$

We substitute the expression for R in Relation (166) and employ Equation (55). Then by Equation (160)

$$\pi_{\omega} \geq \exp \left\{ \left[\frac{|k_{\omega}^2 - 1|}{|k_{\omega}^2 - 1| - \frac{1}{\gamma}} \right] \frac{\pi^2}{6} \right\}, \quad 0 \leq k_{\omega} < k_1 \quad (167)$$

Relations (155) and (167) establish Relation (113).